



Separated and complete adelic models for one-dimensional Noetherian tensor-triangulated categories



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ABSTRACT

We prove the existence of various adelic-style models for rigidly small-generated tensor-triangulated categories whose Balmer spectrum is a one-dimensional Noetherian topological space. This special case of our general programme of giving adelic models is particularly concrete and accessible, and we illustrate it with examples from algebra, geometry, topology and representation theory.

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Contents

1. Introduction	2
2. The adelic framework	5
Part 1. Homotopical models	10
3. The adelic model	10
4. The separated and complete models	13
Part 2. Algebraic models	19
5. Skeletons and the roadmap	19
6. Local skeletons of the nub	20
7. The adelic, separated and L-complete categories	22
8. The cellular skeletons	26
9. The skeletal Quillen equivalences	29
Part 3. Examples	33
10. Abelian groups	33
11. Quasi-coherent sheaves on a curve	35
12. Stable modules for the Klein 4-group	36
13. T-equivariant rational stable homotopy theory	39
References	41

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1. Introduction

In this paper we are concerned with triangulated categories \mathbb{T} equipped with a compatible symmetric monoidal structure $(\otimes, \mathbb{1})$ forming a tensor-triangulated category in the sense of [5]. Our particular focus is on those tensor-triangulated categories which have the formal properties of the derived category of a 1-dimensional commutative Noetherian domain or an irreducible curve. The irreducibility assumption is inessential, and made only to remove distractions, whereas the Noetherian assumption is essential for the simplicity of our approach. Many interesting examples are covered by this hypothesis, some of which are discussed in [Part 3](#).

1.A. Adelic and algebraic models

Structural features of tensor-triangulated categories are controlled by the Balmer spectrum [5], which is the counterpart of the Zariski prime spectrum of a commutative ring. Given a prime \mathfrak{p} in the Balmer spectrum of \mathbb{T} , one may define the localization at \mathfrak{p} (denoted $L_{\mathfrak{p}}$) and the completion at \mathfrak{p} (denoted $\Lambda_{\mathfrak{p}}$). These constructions are analogues of those in commutative algebra.

The idea of decompositions reflecting the dimensional filtration of primes is pervasive in commutative algebra and algebraic geometry. The aim is to show that the subquotients in this filtration can be understood in terms of primes of a single dimension. There are decisions to be made about the way the pure strata are described, and how the assembly information is packaged. Since we are restricting ourselves to the 1-dimensional case in this paper there is only one step to the assembly process, often described in terms of recollements. Discussions of this type can be found in [1,9,27], and we plan to return to the higher dimensional case elsewhere.

The adelic models project [3] also follows this general pattern, but has some distinctive features. The word ‘adelic’ arises because it aims to construct models for \mathbb{T} which are built from localized-completed objects, with the assembly information packaged in adèle-like localizations of products. The second distinctive element is the idea of trying to make the building blocks and assembly information as algebraic as possible. In the most favourable circumstances the building blocks are (derived) categories of modules over a commutative ring and the assembly data is described in terms of a diagram of rings.

This project arose from the earlier work of the second author and collaborators on algebraic models for rational G -spectra, starting with the circle group [14,21,6]. In that case, the adelic model could be shown to be formal, hence showing both that the homotopy category is determined by the pieces it is built from and also that the result is algebraic.

It is therefore natural that there are two elements to the adelic project. The first breaks down a tensor-triangulated category in an adelic fashion, and the second studies algebraic examples of this type. The present paper follows this pattern, by firstly establishing that adelic models exist in considerable generality, and secondly showing that in an algebraic context these models may be further simplified. It is in the nature of the goals that our principal examples will be close to algebra (modules over commutative rings, algebraic geometry, representation theory, *rational* G -spectra). Although our homotopical models apply generally, the extent to which they add to alternative approaches will depend on calculational specifics of individual cases.

This paper focuses on the case when the Balmer spectrum is a Noetherian space, where the topology is determined by the Balmer spectrum as a partially ordered set. In the 1-dimensional case this just means the space of closed points has the cofinite topology. Many naturally occurring examples are not Noetherian: ongoing work of the authors with Barthel aims towards constructing models in this more general case via the yoga of spectral spaces [2].

1.B. Objectwise decomposition

The one-dimensional irreducible case of [3, Theorem 8.1] tells us that under some mild hypothesis on \mathbb{T} the unit object can be recovered by the homotopy pullback of ring objects

$$\begin{array}{ccc}
 \mathbb{1} & \longrightarrow & L_{\mathfrak{g}}\mathbb{1} \\
 \downarrow & & \downarrow \\
 \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} & \longrightarrow & L_{\mathfrak{g}}\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} .
 \end{array} \tag{1}$$

Here \mathfrak{g} is the generic point of the Balmer spectrum and \mathfrak{m} ranges over the closed points (i.e., Balmer minimal primes).

When $\mathbb{T} = D(\mathbb{Z})$, with unit the integers \mathbb{Z} , this recovers the classical Hasse square. If \mathbb{T} is the homotopy category of rational \mathbb{T} -spectra (where \mathbb{T} is the circle group) with unit the sphere spectrum \mathbb{S} then this is the Tate square for the family of finite subgroups [18].

The idea is to use this fracturing of the monoidal unit to provide small and computationally advantageous models of \mathbb{T} while also giving an insight into the global structure of the category. The first model related to this fracturing is the *adelic model* of [3]. This model reconstructs an object $X \in \mathbb{T}$ from modules over the rings $L_{\mathfrak{g}}\mathbb{1}, \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1}$ and $L_{\mathfrak{g}}\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1}$. In essence, it reconstructs an object $X \in \mathbb{T}$ using the homotopy pullback square

$$\begin{array}{ccc}
 X & \longrightarrow & L_{\mathfrak{g}}\mathbb{1} \otimes X \\
 \downarrow & & \downarrow \\
 \left[\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \right] \otimes X & \longrightarrow & \left[L_{\mathfrak{g}}\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \right] \otimes X
 \end{array} \tag{2}$$

obtained as the tensor product of Diagram (1) with X .

This result goes a long way towards reconstructing \mathbb{T} out of categories of modules over localized-complete rings. However, we note that modules over the product ring $\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1}$ can be complicated. The first aim of this paper is to replace this single module category by the separate module categories over the simpler rings $\Lambda_{\mathfrak{m}}\mathbb{1}$, and we refer to this as the *separated model*. In essence, it reconstructs an object $X \in \mathbb{T}$ using the homotopy pullback square

$$\begin{array}{ccc}
 X & \longrightarrow & L_{\mathfrak{g}}\mathbb{1} \otimes X \\
 \downarrow & & \downarrow \\
 \prod_{\mathfrak{m}} [\Lambda_{\mathfrak{m}}\mathbb{1} \otimes X] & \longrightarrow & L_{\mathfrak{g}}\prod_{\mathfrak{m}} [\Lambda_{\mathfrak{m}}\mathbb{1} \otimes X] .
 \end{array} \tag{3}$$

Secondly, one might hope to reconstruct X from its actual completions, which are often more economical, and we will also construct a *complete model* of this type. In essence, it reconstructs an object $X \in \mathbb{T}$ using the homotopy pullback square

$$\begin{array}{ccc}
 X & \longrightarrow & L_{\mathfrak{g}}\mathbb{1} \otimes X \\
 \downarrow & & \downarrow \\
 \prod_{\mathfrak{m}} [\Lambda_{\mathfrak{m}}X] & \longrightarrow & L_{\mathfrak{g}}\prod_{\mathfrak{m}} [\Lambda_{\mathfrak{m}}X] .
 \end{array} \tag{4}$$

Putting it all together, we have a ladder in which all the horizontal fibres are equivalent. This means all subrectangles are homotopy pullback squares, and in particular the three involving X at the top left are the three discussed above.

$$\begin{array}{ccc}
 X & \longrightarrow & L_{\mathfrak{g}}\mathbb{1} \otimes X \\
 \downarrow & & \downarrow \\
 (\prod_m \Lambda_m \mathbb{1}) \otimes X & \longrightarrow & L_{\mathfrak{g}}(\prod_m \Lambda_m \mathbb{1}) \otimes X \\
 \downarrow & & \downarrow \\
 \prod_m (\Lambda_m \mathbb{1} \otimes X) & \longrightarrow & L_{\mathfrak{g}} \prod_m (\Lambda_m \mathbb{1} \otimes X) \\
 \downarrow & & \downarrow \\
 \prod_m (\Lambda_m X) & \longrightarrow & L_{\mathfrak{g}} \prod_m (\Lambda_m X)
 \end{array}$$

Remark 1.1. By taking horizontal fibres, one can rebuild arbitrary objects from local and torsion data. Promoting this to a categorical decomposition is undertaken in [4], which retrieves the torsion model of the second author from [14]. As all of the horizontal fibres coincide, we see that up to homotopy there is only one notion of a torsion model, and the only difference occurs in which category we wish to splice over.

1.C. Homotopical and algebraic models

The point of the paper is to build models based on these objectwise decompositions. We describe these in detail in Section 2, but sketch it here. They all take the form of a diagram

$$\begin{array}{ccc}
 & & V \\
 & & \downarrow \\
 N & \longrightarrow & Q
 \end{array} \tag{5}$$

of modules over the diagram of rings

$$\begin{array}{ccc}
 & & L_{\mathfrak{g}}\mathbb{1} \\
 & & \downarrow \\
 \prod_m \Lambda_m \mathbb{1} & \longrightarrow & L_{\mathfrak{g}} \prod_m \Lambda_m \mathbb{1}
 \end{array} \tag{6}$$

obtained from Diagram (1) by omitting the unit object. The homotopical models simply consist of suitable model structures on the category of diagrams. The adelic model requires cofibrant objects to have the horizontal and vertical base change maps weak equivalences. Accordingly, following [14], we think of the object as the object N (the *nub*) together with the additional structure of V (the *vertex*) and the splicing embodied by maps to Q . The separated model requires in addition that the object N is weakly equivalent to the product of its idempotent pieces, and slightly weakens the condition on the vertical base change map. The complete model requires that N is equivalent to a product and each idempotent piece is homotopically complete, and there is no condition on the vertical base change map. The existence of these homotopical models appears as Theorem 4.1.

In an algebraic context we can go further in each of these three cases. In each case (adelic, separated or complete) the condition on cofibrant objects requires certain base change maps to be weak equivalences. We may consider the subcategory of objects in which these maps are not just equivalences but isomorphisms. Under suitable conditions this category is well behaved and admits a model structure by restriction: the subcategory is said to be a *skeleton* and is Quillen equivalent to the homotopical model. The algebraic models are the content of Corollary 9.2 and Theorems 9.5 and 9.8.

Altogether then we have three homotopical models (adelic, separated and complete), and in well behaved algebraic contexts each of these has a skeleton, giving six models.

1.D. Outline of paper

In Section 2 we recall the apparatus of completions, localizations and diagram categories from [3], and recall the general Hasse pullback square for the unit object.

The remainder of the paper is divided into three parts. In Part 1 (Sections 3 and 4), we focus on models which apply rather generally: they are homotopical in the sense that the important subcategories are identified by homotopical conditions. That is, we explore the theory hinted at in 1.C above.

In Part 2 (Sections 5 to 9), we focus on examples of an algebraic nature where we can identify much smaller subcategories specified by strict conditions, providing skeletons.

Finally, in Part 3 (Sections 10 to 13) we explicitly illustrate the theory for commutative rings, quasi-coherent sheaves over a curve, the stable module category for the Klein 4-group and rational circle-equivariant spectra.

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2. The adelic framework

In this section we briefly recall the relevant machinery from [3] that we will use throughout.

2.A. Noetherian model categories

We will work in the setting of Quillen model categories. We are interested in the homotopy categories of model categories \mathcal{C} which are both stable and monoidal. We continue with the conventions of [4] where the word ‘compact’ refers to the model category and ‘small’ to the homotopy category. We assume the homotopy category $\mathrm{Ho}(\mathcal{C})$ is a rigidly small-generated tensor-triangulated category [24,26]. We write $\mathrm{Ho}(\mathcal{C})^\omega$ for the full subcategory of small objects, our assumptions guarantee that these objects coincide with the dualizable objects.

We denote by $\mathcal{G} \subseteq \mathrm{Ho}(\mathcal{C})^\omega$ a set of compact objects which generate the category $\mathrm{Ho}(\mathcal{C})$. In particular an object $X \in \mathrm{Ho}(\mathcal{C})$ is zero if and only if $[g, X]_* = 0$ for every $g \in \mathcal{G}$.

In this context we may introduce the main organizational principle from [5].

Definition 2.1. A *prime ideal* in a tensor-triangulated category is a proper thick tensor ideal \mathfrak{p} with the property that $a \otimes b \in \mathfrak{p}$ implies that a or b is in \mathfrak{p} . The *Balmer spectrum* of a tensor-triangulated category $\mathrm{Ho}(\mathcal{C})$ is the set of prime tensor ideals of the triangulated category of compact objects:

$$\mathrm{Spc}^\omega(\mathrm{Ho}(\mathcal{C})) = \{\mathfrak{p} \subseteq \mathrm{Ho}(\mathcal{C})^\omega \mid \mathfrak{p} \text{ is prime}\}.$$

We will restrict attention to Balmer spectra that are Noetherian in the sense that chains of open sets satisfy the ascending chain condition. Noetherian spectral spaces enjoy a simple description.

Lemma 2.2. *Let X be a Noetherian spectral space. The closed subsets of X are precisely the finite unions of the closures of points. In particular, the topology on X is determined by the poset structure given by the specialization order.*

As in [3, §4], a *Noetherian model category* is a cellular, proper, stable, monoidal model category \mathcal{C} such that $\text{Ho}(\mathcal{C})$ is a rigidly small-generated tensor-triangulated category whose Balmer spectrum is a Noetherian topological space.

We say that a prime \mathfrak{p} is *visible* if there is a small object $K_{\mathfrak{p}}$ so that $\overline{\{\mathfrak{p}\}} = \text{supp}(K_{\mathfrak{p}}) = \{\mathfrak{q} \in \text{Spc}^{\omega}(\text{Ho}(\mathcal{C})) \mid K_{\mathfrak{p}} \notin \mathfrak{q}\}$. In light of Lemma 2.2, we have the following useful characterization of Noetherian spectral spaces.

Lemma 2.3 ([5, Corollary 2.17]). *The topological space $\text{Spc}^{\omega}(\text{Ho}(\mathcal{C}))$ is Noetherian if and only if all primes are visible.*

2.B. Localizations and completions

Localization and completion are central to our approach, and we summarize material from [3, §5,6]. We freely use results about left and right Bousfield localization from [23].

Let \mathfrak{p} be an arbitrary Balmer prime. The localization at \mathfrak{p} , denoted $L_{\mathfrak{p}}$, is the nullification of the thick tensor subcategory \mathfrak{p} . By assumption, the objects of \mathfrak{p} are all small, and as such $L_{\mathfrak{p}}$ is a finite localization, and therefore is smashing in the sense that there is a weak equivalence $L_{\mathfrak{p}}X \simeq L_{\mathfrak{p}}\mathbb{1} \otimes X$.

For a Balmer prime \mathfrak{p} , we choose a small object $K_{\mathfrak{p}}$ such that $\text{supp}(K_{\mathfrak{p}}) = \overline{\{\mathfrak{p}\}}$, which exists by the assumption that the Balmer spectrum is Noetherian. Any two generate the same thick tensor ideal so subsequent constructions do not depend on the choice. The completion at \mathfrak{p} , denoted $\Lambda_{\mathfrak{p}}$ is then the Bousfield localization with respect to the object $K_{\mathfrak{p}}$. In particular we shall say that a map $f: X \rightarrow Y$ is a weak equivalence in $\Lambda_{\mathfrak{p}}\mathcal{C}$ if $f \otimes K_{\mathfrak{p}}: X \otimes K_{\mathfrak{p}} \rightarrow Y \otimes K_{\mathfrak{p}}$ is a weak equivalence in \mathcal{C} . Unlike the $L_{\mathfrak{p}}$, these completions are not usually smashing, and do not usually preserve small objects. Both of these Bousfield localizations exist for Noetherian model categories and are stable monoidal localizations [3, §5,6].

We also need to consider cellularization (right Bousfield localization) with respect to the thick tensor ideal generated by $K_{\mathfrak{p}}$, giving a $K_{\mathfrak{p}} \otimes \mathcal{G}$ -equivalence $\Gamma_{\mathfrak{p}}X \rightarrow X$ from a $K_{\mathfrak{p}} \otimes \mathcal{G}$ -cellular object. This gives a useful equivalence $\Lambda_{\mathfrak{p}}X \simeq \text{Hom}(\Gamma_{\mathfrak{p}}\mathbb{1}, X)$.

For the rest of the article we will assume in addition that the Balmer spectrum is 1-dimensional and irreducible. This means it has a single generic point \mathfrak{g} (the Balmer-maximal prime) and a set of closed points \mathfrak{m} (Balmer-minimal primes).

Remark 2.4. The geometry of completions and localizations suggests highlighting the downward cone for completion and cellularization ($\Lambda_{\mathfrak{p}} = \Lambda_{\wedge(\mathfrak{p})}$ and $\Gamma_{\mathfrak{p}} = \Gamma_{\wedge(\mathfrak{p})}$) and the upward cone for localization ($L_{\mathfrak{p}} = L_{\vee(\mathfrak{p})}$). For brevity we will retain the simpler traditional notation.

The main result from [3] tells us that we can retrieve the monoidal unit $\mathbb{1}$ from its localized completions. All of the models appearing in this paper will come from various interpretations of this result.

Theorem 2.5 (*Adelic Approximation Theorem*). *Let \mathcal{C} be a 1-dimensional Noetherian model category whose Balmer spectrum is one-dimensional and irreducible. Then the unit object $\mathbb{1}$ is the homotopy pullback of the following diagram of rings in \mathcal{C}*

$$\mathbb{1}_{ad} := \left[\begin{array}{ccc} & & L_{\mathfrak{g}} \mathbb{1} \\ & & \downarrow j \\ \prod_m \Lambda_m \mathbb{1} & \xrightarrow{L_{\mathfrak{g}}} & L_{\mathfrak{g}} \prod_m \Lambda_m \mathbb{1} \end{array} \right]$$

Proof. One takes the horizontal fibre and sees that the map $\Gamma_{\mathfrak{g}} \mathbb{1} \rightarrow \Gamma_{\mathfrak{g}} \prod_m \Lambda_m \mathbb{1}$ is a weak equivalence since $\mathbb{1} \rightarrow \prod_m \Lambda_m \mathbb{1}$ is a K_m equivalence for all m . This is the one-dimensional version of the general result [3, Theorem 8.1] and a special case of the Tate square [15, Corollary 2.4]. \square

2.C. Diagrams of model categories

Diagrams of model categories are at the core of our method. The only diagram shape that we will need to consider is that of a cospan:

$$\alpha = \left(\begin{array}{ccc} & & \mathcal{C}_1 \\ & & \downarrow F_1 \\ \mathcal{C}_2 & \xrightarrow{F_2} & \mathcal{C}_3 \end{array} \right)$$

A generalized diagram over this cospan is displayed as

$$\begin{array}{ccc} & & c_1 \\ & & \vdots \\ & & f_1 \\ & & \downarrow \\ c_2 & \xrightarrow{f_2} & c_3 \end{array}$$

More explicitly, it is a quintuple

$$(c_1, c_2, c_3, f_1: F_1 c_1 \rightarrow c_3, f_2: F_2 c_2 \rightarrow c_3)$$

where $c_i \in \mathcal{C}_i$. A morphism

$$(c_1, c_2, c_3, f_1: F_1 c_1 \rightarrow c_3, f_2: F_2 c_2 \rightarrow c_3) \rightarrow (c'_1, c'_2, c'_3, f'_1: F_1 c'_1 \rightarrow c'_3, f'_2: F_2 c'_2 \rightarrow c'_3)$$

is given by morphisms $\alpha_i: c_i \rightarrow c'_i$ in \mathcal{C}_i for $i = 0, 1, 2$ such that the squares in the following diagram commute:

$$\begin{array}{ccccc} F_1 c_1 & \xrightarrow{f_1} & c_3 & \xleftarrow{f_2} & F_2 c_2 \\ F_1 \alpha_1 \downarrow & & \alpha_3 \downarrow & & \downarrow F_2 \alpha_2 \\ F_1 c'_1 & \xrightarrow{f'_1} & c'_3 & \xleftarrow{f'_2} & F_2 c'_2 \end{array}$$

These diagrams and morphisms between them form a category: we think of the categories \mathcal{C}_i as fibres over the points of the diagram shape so that the (generalized) diagram is a *section*, and write

$$\mathcal{C}_\alpha = \mathbb{I} \left(\begin{array}{ccc} & & \mathcal{C}_1 \\ & & \downarrow F_1 \\ \mathcal{C}_2 & \xrightarrow{F_2} & \mathcal{C}_3 \end{array} \right)$$

for the category of sections.

If we further assume that the categories \mathcal{C}_i come equipped with model structures, and that the functors F_1, F_2 are left Quillen functors then the category of sections admits the injective model structure in which weak equivalences and cofibrations are objectwise [20, Theorem 3.1]. The injective model structure inherits many properties that all the \mathcal{C}_i have, such as being proper and cellular. We note that this injective model structure is the lax homotopy limit of the diagram in the sense of [13].

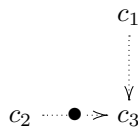
2.D. Quasi-coherence and extendedness

In our situation, the functor F_2 appearing in the diagram category will have the character of a localization, and the vertical functor F_1 will have the flavour of a faithfully flat extension.

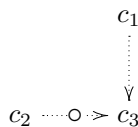
Definition 2.6. Let $X = (c_1, c_2, c_3, f_1: F_1c_1 \rightarrow c_3, f_2: F_2c_2 \rightarrow c_3)$ be an object of \mathcal{C}_α . We say that X is:

- *quasi-coherent* (qc) if the map $F_2c_2 \rightarrow c_3$ is an isomorphism in \mathcal{C}_3 .
- *extended* (e) if the map $F_1c_1 \rightarrow c_3$ is an isomorphism in \mathcal{C}_3 .
- if we only require weak equivalences in the injective model structure, we say that X is *weakly quasi-coherent* (wqc) or *weakly extended* (we). We write wqce for the class of weakly quasi-coherent, weakly extended objects.

We find it helpful to write



to indicate a quasi-coherent object and



to indicate a weakly quasi-coherent object. Similar notation applies for (weakly) extended and (weakly) quasi-coherent extended objects, as well as for subcategories of such objects.

2.E. Homotopical models

Having described the diagrammatic framework, we can outline how to lift the decomposition on objects to a categorical level. Full details will be given in Sections 3 and 4.

We assume that \mathbb{T} is the homotopy category of some stable monoidal model category \mathcal{C} with 1-dimensional, irreducible, Noetherian Balmer spectrum, and consider the diagram of rings in Display (6). The adelic category \mathcal{C}_{ad} consists of modules over this diagram of rings. Our homotopical models will be formed from

\mathcal{C}_{ad} by Bousfield localization. For a model category \mathcal{D} , we write $R_A\mathcal{D}$ for the cellularization with respect to the thick tensor ideal generated by the set A of objects of \mathcal{D} , and for an object E of \mathcal{D} we write $L_E\mathcal{D}$ for the left localization inverting E -homology isomorphisms.

One of the main results of [3] states that \mathcal{C} is Quillen equivalent to a model in which the bifibrant objects are the weakly quasi-coherent extended (wqce) objects in \mathcal{C}_{ad} . These are exactly the objects which are reconstructed as in Diagram (2). More precisely, we give \mathcal{C}_{ad} the diagram injective model structure and then form the cellularization $R_{wqce}\mathcal{C}_{ad}$ and obtain an equivalence $R_{wqce}\mathcal{C}_{ad} \simeq_Q \mathcal{C}$.

To access the objects arising in the situation of Diagram (3) we begin instead by describing a slightly larger class $wqcie \supseteq wqce$ of objects that treats the different closed points separately. If we use the less extreme localization $R_{wqcie}\mathcal{C}_{ad}$ we do not get a model of \mathcal{C} since too many objects remain bifibrant. To compensate for this we can apply a mild left localization L_Π to ensure that the fibrant objects have the object M equivalent to a product of modules over the factors $\Lambda_m\mathbb{1}$. This leads to a model $L_\Pi R_{wqcie}\mathcal{C} \simeq_Q \mathcal{C}$ which we call the *separated model*. It is homotopical in the sense that the ambient category still has all $\prod_m \Lambda_m\mathbb{1}$ -modules at the nub, and they are only required to be products up to homotopy.

Less extreme still, we can form the even larger class $wqc \supseteq wqcie \supseteq wqce$ of weakly quasi-coherent objects. We then form the cellularization $R_{wqc}\mathcal{C}_{ad}$ with respect to wqc . Once again we do not yet arrive at a model of \mathcal{C} . However, we perform the same balancing act as above, and apply a more extreme left localization L_Λ to obtain one. This localization is taken in such a way to force the fibrant objects to have the module M to have the homotopy type not only a product of modules, but the product of complete modules. That is, such a model is the reconstruction with respect to Diagram (4). As such we call $L_\Lambda R_{wqc}\mathcal{C} \simeq_Q \mathcal{C}$ the *complete model*.

To summarize, we will prove that there are Quillen equivalences

$$\mathcal{C} \simeq_Q R_{wqce}\mathcal{C}_{ad} \simeq_Q L_\Pi R_{wqcie}\mathcal{C}_{ad} \simeq_Q L_\Lambda R_{wqc}\mathcal{C}_{ad}$$

which provides us with three models whose underlying category is \mathcal{C}_{ad} .

2.F. Algebraic models

When the categories in question are of an algebraic nature, we may consider the cellular skeleton, where the underlying category is pared down to the absolute minimum so that the homotopy theory has the least amount of work to do. For example, we consider the full subcategory $\mathcal{C}_{ad}^{qce} \subseteq \mathcal{C}_{ad}$ of objects whose base change maps are isomorphisms (as opposed to being weak equivalences). One can then prove that the restricted model structure on \mathcal{C}_{ad}^{qce} induced from the diagram-injective model on \mathcal{C}_{ad} is Quillen equivalent to $R_{wqce}\mathcal{C}_{ad}$ and hence also to \mathcal{C} .

Similarly, the skeleton of the separated model requires that the nub is actually a product, and a suitably adapted version of the vertical basechange map is an isomorphism. For a skeleton of the complete model, we need the more stringent restriction to an algebraic context where there is an abelian model for complete modules, but in that case we again obtain a skeleton.

2.G. Summary

To summarize, in the most favourable situation we obtain three different models. In this setting, an object $X \in \text{Ho}(\mathcal{C})$ can be described by each of the following three sets of data:

Adelic:

- V a $L_{\mathfrak{g}}\mathbb{1}$ -module;
- N a $\prod_m \Lambda_m\mathbb{1}$ -module;

- An equivalence $L_{\mathfrak{g}}N \simeq j_*V$ of $L_{\mathfrak{g}} \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1}$ -modules.

Separated:

- V a $L_{\mathfrak{g}}\mathbb{1}$ -module;
- $N_{\mathfrak{m}}$ an $\Lambda_{\mathfrak{m}}\mathbb{1}$ -module for each \mathfrak{m} ;
- A map of $j_*V \rightarrow L_{\mathfrak{g}} \prod_{\mathfrak{m}} N_{\mathfrak{m}}$ of $L_{\mathfrak{g}} \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1}$ -modules so that for each idempotent $e_{\mathfrak{m}}$ it gives an equivalence $e_{\mathfrak{m}}j_*V \xrightarrow{\simeq} e_{\mathfrak{m}}L_{\mathfrak{g}}N_{\mathfrak{m}}$.

Complete:

- V a $L_{\mathfrak{g}}\mathbb{1}$ -module;
- $N_{\mathfrak{m}}$ a complete $\Lambda_{\mathfrak{m}}\mathbb{1}$ -module for each \mathfrak{m} ;
- A map $V \rightarrow L_{\mathfrak{g}} \prod_{\mathfrak{m}} N_{\mathfrak{m}}$ of $L_{\mathfrak{g}} \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1}$ -modules.

In Section 10 we make these explicit in the case of the derived category of a 1-dimensional Noetherian domain, and in Section 11 for quasi-coherent sheaves over a curve.

Part 1. Homotopical models

In Part 1 we describe the homotopical adelic, separated and complete models, which apply very generally in the 1-dimensional, Noetherian setting.

3. The adelic model

In this section we recall the adelic model from [3] and prove that this cellularization of the adelic category is a realization of the strict homotopy limit.

3.A. The adelic model as a cellularization

We now have the necessary language to describe the adelic model. We will freely use the technology of module categories in monoidal model categories from [29]. For R a cofibrant monoid in a monoidal model category \mathcal{C} we denote by $R\text{-mod}_{\mathcal{C}}$ the associated category of modules which we equip with the projective model structure.

Definition 3.1. Let \mathcal{C} be a 1-dimensional Noetherian model category. We define the adelic category to be the category of sections of the adelic cospan:

$$\mathcal{C}_{\text{ad}} := \Gamma \left(\begin{array}{ccc} & & (L_{\mathfrak{g}}\mathbb{1})\text{-mod}_{\mathcal{C}} \\ & & \downarrow j_* \\ (\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1})\text{-mod}_{\mathcal{C}} & \xrightarrow{L_{\mathfrak{g}}} & (L_{\mathfrak{g}} \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1})\text{-mod}_{\mathcal{C}} \end{array} \right)$$

The functors in this diagram are extensions of scalars, and therefore left adjoints. In other words, we take the category of modules over the diagram of rings $\mathbb{1}_{\text{ad}}$ from Theorem 2.5.

The categories are related by an adjoint pair

$$\mathcal{C} \begin{array}{c} \xrightarrow{a} \\ \xleftarrow{p} \end{array} \mathcal{C}_{\text{ad}}$$

where a is defined via pointwise tensoring with the diagram $\mathbb{1}_{ad}$, and p is the pullback of the diagram when viewed in \mathcal{C} .

We equip the individual module categories with the module projective model structures, and \mathcal{C}_{ad} with the diagram injective model structure. With these model structures the above adjoint pair is a Quillen pair. One checks that on homotopy categories a preserves small objects, and that the derived unit is a weak equivalence as a consequence of the Adelic Approximation Theorem 2.5 which tells us that $p(a(\mathbb{1})) = p(\mathbb{1}_{ad}) \simeq \mathbb{1}$.

Recall that we have assumed that $\text{Ho}(\mathcal{C})$ has a set \mathcal{G} of small generators. We shall write \mathcal{G}_{ad} for the image of the generators under a . We now apply the cellularization principle [19] to obtain the basic adelic model.

Theorem 3.2. [3, Theorem 9.3] *Let \mathcal{C} be a 1-dimensional Noetherian model category. The above adjunction is a Quillen equivalence*

$$\mathcal{C} \simeq_Q R_{\mathcal{G}_{ad}} \mathcal{C}_{ad}$$

between \mathcal{C} and the cellularization of \mathcal{C}_{ad} at the image of the generators.

3.B. The adelic model as a strict homotopy limit

The cellularization appearing in Theorem 3.2 has an attractive universal property: it coincides with the strict homotopy limit as defined by Bergner and Barwick [10,13] which we will denote $\mathcal{L}\text{im}(\mathcal{C}_{ad})$ ([3, 9.3]). We will show that the subcategory of bifibrant objects in $R_{\mathcal{G}_{ad}} \mathcal{C}_{ad}$ consists of wqce-objects which are object-wise cofibrant. We can describe this subcategory as

$$\mathcal{C}_{ad}^{\text{wqce}} = \mathbb{I} \left(\begin{array}{ccc} & & (L_{\mathfrak{g}} \mathbb{1})\text{-mod}_{\mathcal{C}} \\ & & \downarrow \circ \\ (\prod_m \Lambda_m \mathbb{1})\text{-mod}_{\mathcal{C}} & \dashrightarrow & (L_{\mathfrak{g}} \prod_m \Lambda_m \mathbb{1})\text{-mod}_{\mathcal{C}} \end{array} \right)$$

where we have used the \circ notation as in Section 2.D, and we have suppressed notation to indicate that we are requiring diagrams to be object-wise cofibrant.

We now introduce some special objects of $\mathcal{C}_{ad}^{\text{wqce}}$ by considering the right adjoints to evaluation at a prime. First we consider the generic prime \mathfrak{g} . For an arbitrary $L_{\mathfrak{g}} \mathbb{1}$ -module W , define the object

$$e(W) = \left[\begin{array}{ccc} & W & \\ & \vdots & \\ & \circ & \\ j_* W & \dashrightarrow & j_* W \end{array} \right] \in \mathcal{C}_{ad}^{\text{wqce}}.$$

In the bottom left hand corner notation for restriction of scalars along the ring map $\prod_m \mathbb{1} \rightarrow L_{\mathfrak{g}} \prod \Lambda_m \mathbb{1}$ has been omitted.

An $L_{\mathfrak{g}}$ -module W can be viewed as an object of \mathcal{C} via the forgetful functor to which we can apply the functor a , giving a functor from $L_{\mathfrak{g}}$ -modules to $\mathcal{C}_{ad}^{\text{wqce}}$.

Lemma 3.3. *The object $e(W)$ is equivalent to an object coming from \mathcal{C} in that it is in the image of a . Indeed, $e(W)$ is the image of W itself viewed as a $\mathbb{1}$ -module in \mathcal{C} . The functor $e: (L_{\mathfrak{g}} \mathbb{1})\text{-mod}_{\mathcal{C}} \rightarrow \mathcal{C}_{ad}$ is right adjoint to evaluation at \mathfrak{g} (i.e., at the vertex) on homotopy category of \mathcal{C}_{ad} .*

Proof. By definition

$$a(W) = \left[\begin{array}{ccc} & & L_{\mathfrak{g}}\mathbb{1} \otimes W \\ & & \vdots \\ & & \circ \\ & & \downarrow \\ \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \otimes W & \cdots \circ \triangleright & L_{\mathfrak{g}} \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \otimes W \end{array} \right]$$

Since W is a $L_{\mathfrak{g}}\mathbb{1}$ -module we have $L_{\mathfrak{g}}\mathbb{1} \otimes W \simeq W$, and by definition the right hand vertical functor is extension of scalars. Since the bottom left entry is $L_{\mathfrak{g}}$ -local, and the lower horizontal functor is $L_{\mathfrak{g}}$ -localization it follows that the bottom left entry is also equivalent to j_*W . Hence $a(W) \simeq e(W)$ as required.

For the adjoint property we see

$$\left[\left[\begin{array}{ccc} & & V \\ & & \vdots \\ & & \circ \\ & & \downarrow \\ N & \cdots \circ \triangleright & P \end{array} \right], \left[\begin{array}{ccc} & & W \\ & & \vdots \\ & & \circ \\ & & \downarrow \\ j_*W & \cdots \circ \triangleright & j_*W \end{array} \right] \right] = [V, W]. \quad \square$$

When it comes to minimal primes, there is no right adjoint on the whole subcategory. However, if N is a $(\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1})$ -module which is torsion in the sense that $L_{\mathfrak{g}}N \simeq 0$, then we may take

$$f(N) = \left[\begin{array}{ccc} & & 0 \\ & & \vdots \\ & & \circ \\ & & \downarrow \\ N & \cdots \circ \triangleright & 0 \end{array} \right] \in \mathcal{C}_{\text{ad}}^{\text{wqce}}$$

Lemma 3.4. *When $L_{\mathfrak{g}}N \simeq 0$, the object $f(N)$ is equivalent to an object in the image of a . Indeed, $f(N)$ is the image of N itself viewed as a $\mathbb{1}$ -module. The object $f(-)$ has the property $\text{Hom}(f(M), f(N)) = \text{Hom}(M, N)$ and $[f(M), f(N)] = [M, N]$.*

Proof. We may calculate

$$a(N) = \left[\begin{array}{ccc} & & L_{\mathfrak{g}}N \\ & & \vdots \\ & & \circ \\ & & \downarrow \\ \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \otimes N & \cdots \circ \triangleright & j_*L_{\mathfrak{g}}N \end{array} \right]$$

Since $L_{\mathfrak{g}}N \simeq 0$, we have that $a(N) \simeq f(\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \otimes N)$. As $\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \otimes N \simeq N$ the result follows as the object is weakly extended.

The statement about maps between $f(M)$ and $f(N)$ is clear due to the fact that the diagrams are concentrated at a single vertex. \square

Remark 3.5. These $f(-)$ and $e(-)$ objects are very useful since they correspond to two layers given by the dimension filtration. That is, the objects $e(-)$ are ones supported on the prime \mathfrak{g} , while the $f(-)$ are supported on the closed points \mathfrak{m} . In particular, these objects build all wqce objects. Indeed, for an arbitrary object

$$X = \left[\begin{array}{ccc} & & V \\ & & \vdots \\ & & \circ \\ & & \downarrow \\ N & \cdots \circ \triangleright & P \end{array} \right] \in \mathcal{C}_{\text{ad}}^{\text{wqce}}$$

there is a natural map $X \rightarrow e(V)$ corresponding to the identity on V and this gives a cofibre sequence

$$f(N') \rightarrow X \rightarrow e(V), \tag{7}$$

where $N' = \text{fibre}(N \rightarrow L_{\mathfrak{g}}N)$.

Theorem 3.6. [3, 9.C] *Every object of $R_{\mathcal{G}_{\text{ad}}}\mathcal{C}_{\text{ad}}$ is equivalent to an object of $\mathcal{C}_{\text{ad}}^{\text{wqce}}$ and hence*

$$\mathcal{C} \simeq_Q R_{\mathcal{G}_{\text{ad}}}\mathcal{C}_{\text{ad}} \simeq_Q \mathcal{L}\text{im}(\mathcal{C}_{\text{ad}}).$$

Proof. By [13, 3.2] the strict homotopy limit is the cellularization of the diagram category with respect to wqce. We have right Bousfield localized with respect to the apparently smaller subcategory of objects coming from \mathcal{C} , so it suffices to show they generate the same subcategory.

Suppose then that $X \in \mathcal{C}_{\text{ad}}^{\text{wqce}}$, and use the notation above for its values. We must show that X is trivial in \mathcal{C}_{ad} if and only if it is cellularly trivial. It is obvious that if X is trivial then it is cellularly trivial. Suppose then that X is cellularly trivial in the sense that $[a(g), X]_* = 0$ for all $g \in \mathcal{G}$. It follows that $[a(A), X]_* = 0$ for all $A \in \mathcal{C}$.

Now consider the cofibre sequence (7). Since $[f(N'), e(V)]_* = 0$ we see that

$$[N', N']_* = [f(N'), f(N')]_* = [f(N'), X]_* = 0$$

so $N' \simeq 0$ and it follows that $X \simeq e(V)$. But now

$$[V, V]_* = [e(V), e(V)]_* = [e(V), X]_* = 0$$

so that $V \simeq 0$. Hence $X \simeq 0$ as required. \square

Remark 3.7. Theorem 3.6 shows that \mathcal{G}_{ad} generates the localizing subcategory wqce. Since \mathcal{G}_{ad} is a set of small objects, it shows that cellularization with respect to wqce exists and $R_{\text{wqce}}\mathcal{C}_{\text{ad}} = R_{\mathcal{G}_{\text{ad}}}\mathcal{C}_{\text{ad}}$. To avoid referring to any particular generators, we henceforth write $R_{\text{wqce}}\mathcal{C}_{\text{ad}}$ for the adelic model of \mathcal{C} .

4. The separated and complete models

In this section we construct both the separated and complete models from \mathcal{C}_{ad} by taking a left localization of a cellularization of \mathcal{C}_{ad} . We will introduce the relevant localizations and then prove the following theorem.

Theorem 4.1 (Adelic, separated and complete models). *Let \mathcal{C} be a 1-dimensional Noetherian model category. Then there are Quillen equivalences*

$$\mathcal{C} \simeq_Q R_{\text{wqce}}\mathcal{C}_{\text{ad}} \simeq_Q L_{\Pi}R_{\text{wqce}}\mathcal{C}_{\text{ad}} \simeq_Q L_{\Lambda}R_{\text{wqc}}\mathcal{C}_{\text{ad}}.$$

The outline for this section is as follows. In Subsection 4.A we will introduce the relevant classes for cellularization, in Subsection 4.B we introduce the left localizations. Subsection 4.C details the strategy, and then in Subsection 4.D we assemble the pieces to prove the theorem.

4.A. Classes of cofibrant objects

We introduce three classes of objects which will be classes of cofibrant objects in our models. To show they can be the class of cofibrant objects we need to establish they are generated by a set of small objects. All

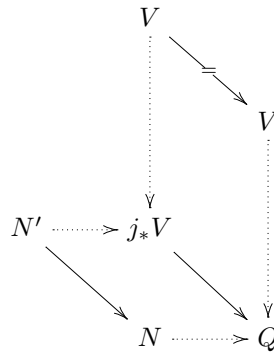
are weakly quasi-coherent (wqc, a condition on the horizontal base change) but then we require additional conditions on the vertical base change (either extendedness or the somewhat weaker condition of idempotent extendedness). This gives the hierarchy

$$\text{wqc} \supseteq \text{wqcie} \supseteq \text{wqce}.$$

4.A.1. Weakly quasi-coherent objects

In Section 3.B we proved that the class of wqc objects was generated by the set \mathcal{G}_{ad} (see Remark 3.7). We now prove wqc is also generated by a set of small objects. Since we are working stably, it is enough to show that the acyclic objects in the cellularization are generated by a set of small objects.

We suppose $N \dashrightarrow Q \dashleftarrow V$ is wqc (i.e., $L_{\mathfrak{g}}N \simeq Q$) and then form



where N' is constructed as the pullback. As we have assumed that \mathcal{C} (and hence $(L_{\mathfrak{g}}\mathbb{1})\text{-mod}_{\mathcal{C}}$) is right proper, it follows that $L_{\mathfrak{g}}N' \simeq j_*V$. In particular, the back face is a wqc object. Therefore the mapping cone of this morphism of diagrams is of the form $M \dashrightarrow P \dashleftarrow 0$ where $P \simeq L_{\mathfrak{g}}M$. It remains to observe any such object can be \otimes -built from the small wqc object $\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \dashrightarrow L_{\mathfrak{g}}\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \dashleftarrow 0$.

4.A.2. Modules over product rings

We are considering modules M over product rings $\prod_i R_i$. Using the projection $p_i : \prod_i R_i \rightarrow R_i$ we may define an R_i -module by extension of scalars

$$e_iM := R_i \otimes_{\prod_i R_i} M.$$

This is idempotent in the sense that if we view e_iM as an $\prod_i R_i$ -module by restriction $e_i e_i M \cong e_i M$ since $R_i \otimes_{\prod_i R_i} R_i \cong R_i$.

Indeed, there is an adjunction

$$(\prod_i R_i)\text{-mod} \begin{matrix} \xrightarrow{\sigma} \\ \xleftarrow{\pi} \end{matrix} \prod_i (R_i\text{-mod})$$

The right adjoint π simply takes products of modules, and the left adjoint σ is defined by $\sigma(M)_i = e_i M$. For a module M over $\prod_i R_i$ we write $M^{\Pi} = \pi\sigma M$, and think of the unit of the adjunction $M \rightarrow M^{\Pi}$ as a form of completion.

4.A.3. Weakly quasi-coherent idempotent extended objects

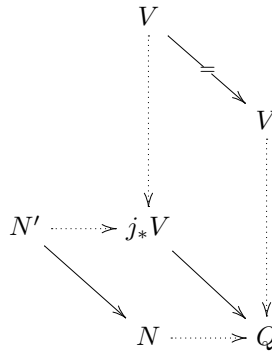
We now consider the special case of product rings where the factors are the completions $\Lambda_{\mathfrak{m}}\mathbb{1}$.

Definition 4.2. A map $V \rightarrow Q$ of modules over $L_{\mathfrak{g}}\mathbb{1} \rightarrow L_{\mathfrak{g}}\prod_{\mathfrak{m}}\Lambda_{\mathfrak{m}}\mathbb{1}$ is *idempotent extended* if the idempotent pieces $e_{\mathfrak{m}}j_*V \xrightarrow{\cong} e_{\mathfrak{m}}Q$ of the base change map are isomorphisms for all closed points \mathfrak{m} . It is *weakly idempotent extended* if the maps are weak equivalences, and we write $wqcie$ for the collection of all weakly quasi coherent, weakly idempotent extended objects.

Since passage to idempotent pieces preserves weak equivalences, a weakly extended object is weakly idempotent extended and $wqcie \supseteq wqce$.

By similar arguments to Section 3.B and 4.A.1 we see that $wqcie$ is the class of cofibrant objects in suitable cellularizations with respect to a set of small objects.

We suppose $N \dashrightarrow Q \dashleftarrow V$ is $wqcie$ and then once again form



As before, the mapping cone is of the form $M \dashrightarrow P \dashleftarrow 0$ where $P \simeq L_{\mathfrak{g}}M$, and we also see that M is Π -trivial in the sense that M is not necessarily equivalent to 0, but $M^{\Pi} \simeq 0$.

It then remains to observe that any Π -trivial object is \otimes -built from the Π -trivial object $T := \prod_{\mathfrak{m}}\Lambda_{\mathfrak{m}}\mathbb{1}/\bigoplus_{\mathfrak{m}}\Lambda_{\mathfrak{m}}\mathbb{1}$. This may not be small for all modules but $[T, \bigoplus_i N_i] = \bigoplus_i [T, N_i]$ if N_i is Π -trivial since $[T, N] = [\prod_{\mathfrak{m}}\Lambda_{\mathfrak{m}}\mathbb{1}, N]$ when N is Π -trivial.

4.B. Bousfield localizations

We now explore the appropriate left Bousfield localizations corresponding to the right Bousfield localizations R_{wqce}, R_{qcie} and R_{wqc} as in Subsection 2.E.

Given an object $E = (N \dashrightarrow Q \dashleftarrow V)$ of \mathcal{C}_{ad} , we may consider the set E -we of those maps f so that $E \otimes f$ is a weak equivalence. We note that this depends only on the objects V, Q and N appearing in the diagram and not on the base-change maps between them. Since we work stably, it is enough to consider the set $\langle E \rangle$ of E -acyclics.

4.B.1. The Π Bousfield localization

We first consider a localization designed to ensure the bottom left object is a product of modules for the individual factors $\Lambda_{\mathfrak{m}}\mathbb{1}$, to be used for the separated model. This will be Bousfield localization with respect to

$$\Pi = \{ \Lambda_{\mathfrak{m}}\mathbb{1} \dashrightarrow L_{\mathfrak{g}}\Lambda_{\mathfrak{m}}\mathbb{1} \dashleftarrow L_{\mathfrak{g}}\mathbb{1} \mid \mathfrak{m} \text{ a closed point} \}$$

We remark that the $wqce$ -cofibrant replacement is

$$\Gamma_{wqce}(\Pi) \simeq \{ \Lambda_{\mathfrak{m}}\mathbb{1} \times L_{\mathfrak{g}}\Lambda_{\mathfrak{m}'}\mathbb{1} \dashrightarrow L_{\mathfrak{g}}\prod_{\mathfrak{m}}\Lambda_{\mathfrak{m}}\mathbb{1} \dashleftarrow L_{\mathfrak{g}}\mathbb{1} \mid \mathfrak{m} \text{ a closed point} \},$$

where $\Lambda_{\mathfrak{m}'}\mathbb{1} = \prod_{\mathfrak{n} \neq \mathfrak{m}}\Lambda_{\mathfrak{n}}\mathbb{1}$.

4.B.2. The Λ Bousfield localization

Now we consider a localization designed to ensure the bottom left object is a product of complete modules for the individual factors $\Lambda_{\mathfrak{m}}\mathbb{1}$, to be used in the complete model. This will be Bousfield localization with respect to

$$\Lambda = \{ K_{\mathfrak{m}} \dashrightarrow 0 \leftarrow L_{\mathfrak{g}}\mathbb{1} \mid \mathfrak{m} \text{ a closed point} \}$$

where $K_{\mathfrak{m}}$ is a Koszul object with support $\overline{\{\mathfrak{m}\}}$ as used to define completion.

We remark that the wqce-cofibrant replacement is

$$\Gamma_{\text{wqce}}(\Lambda) \simeq \{ K_{\mathfrak{m}} \times L_{\mathfrak{g}} \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \dashrightarrow L_{\mathfrak{g}} \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1} \leftarrow L_{\mathfrak{g}}\mathbb{1} \mid \mathfrak{m} \text{ a closed point} \}.$$

It is clear that the set Π generates a localizing subcategory containing Λ . As such it follows that $L_{\Pi} = L_{\Pi \amalg \Lambda}$, and there is a natural transformation $L_{\Pi} \rightarrow L_{\Lambda}$.

4.C. Quillen equivalences of left and right localizations

In this subsection we describe the main strategy of proof for Theorem 4.1: we provide sufficient conditions for various composites of left and right Bousfield localizations to be Quillen equivalences. In the next section we will apply these criteria to establish the separated and complete models.

In fact we will be interested in zig-zags of left Quillen functors

$$\mathcal{C} \leftarrow RC \rightarrow LRC.$$

This zig-zag is obtained by considering the square

$$\begin{array}{ccc} \mathcal{C} & \longleftarrow & RC \\ \downarrow & & \downarrow \\ LC & \longleftarrow & LRC \end{array} .$$

We will give criteria for these localizations to be Quillen equivalences.

Recall that a left localization $\mathcal{C} \rightarrow LC$ is a Quillen equivalence if all the maps that are inverted are already weak equivalences. We observe that it is enough to check this on weak equivalences between cofibrant objects, and the cofibrant objects are the same in both categories. Once again, for stable localizations it is enough to check this for acyclics. That is, we must show that if x is a cofibrant object with $x \simeq 0$ in LC then in fact $x \simeq 0$ in \mathcal{C} .

Similarly to show that a right localization $RD \rightarrow \mathcal{D}$ is a Quillen equivalence we must show that if $x \rightarrow y$ is a cellular equivalence then it is an actual equivalence. It is also sufficient to do this for fibrant objects, and fibrant objects are the same in both categories. In the stable setting it is enough to do it for acyclics. That is, we must show that if x is a fibrant object with $x \simeq 0$ in RD then in fact $x \simeq 0$ in \mathcal{D} .

Of particular interest to this section will be those left Quillen functors of the form $LRD \rightarrow LD$. Suppose we have a morphism $f: x \rightarrow y$ where x is cofibrant in LRD and y is fibrant in LD . We must then show that f is a weak equivalence in LRD if and only if it is a weak equivalence in LD . We now assume that L is a homological localization with respect to some object E . Observe that it is enough to check for x cofibrant in RD and y fibrant in LD that if f is an equivalence in LRD then it is an equivalence in LD . In particular, it suffices to show that if $E \otimes x \rightarrow E \otimes y$ is an equivalence in RD then it is an equivalence in \mathcal{D} .

4.D. Cases at hand

For the cofibrant objects we have three candidates $wqce \subseteq wqcie \subseteq wqc$. We showed in Sections 3.B, 4.A.1 and 4.A.3 that each of these three classes is the sets of cofibrant objects in the cellularization with respect to a set of small objects. Accordingly, we can form right Bousfield localizations with respect to each of them.

For controlling the fibrant objects we wish to use the left Bousfield localizations introduced in Section 4.B, namely L_Π and L_Λ .

Assembling the pieces, one may construct the following diagram of left Quillen functors, where the marked equivalences are the things we shall now prove:

$$\begin{array}{ccccccc}
 \mathcal{C}_{ad} & \longleftarrow & R_{wqc}\mathcal{C}_{ad} & \longleftarrow & R_{wqcie}\mathcal{C}_{ad} & \longleftarrow & R_{wqce}\mathcal{C}_{ad} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 L_\Pi\mathcal{C}_{ad} & \longleftarrow & L_\Pi R_{wqc}\mathcal{C}_{ad} & \longleftarrow & L_\Pi R_{wqcie}\mathcal{C}_{ad} & \xleftarrow{\simeq} & L_\Pi R_{wqce}\mathcal{C}_{ad} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 L_\Lambda\mathcal{C}_{ad} & \longleftarrow & L_\Lambda R_{wqc}\mathcal{C}_{ad} & \xleftarrow{\simeq} & L_\Lambda R_{wqcie}\mathcal{C}_{ad} & \xleftarrow{\simeq} & L_\Lambda R_{wqce}\mathcal{C}_{ad}
 \end{array}$$

In the following discussion we will as usual have an object

$$x = \left(\begin{array}{c} V \\ \vdots \\ N \longrightarrow Q \end{array} \right)$$

with nub N , and vertex V .

To establish the right hand vertical equivalences, since $\Lambda \subseteq \Pi$, it suffices to prove that if a $wqce$ object is Λ -trivial then it is trivial. Suppose then that x is $wqce$ and moreover Λ -trivial. Since the vertex V is at an initial point of the diagram, we conclude that $V \simeq 0$. Since the object x is weakly extended we have $j_*V \simeq Q \simeq 0$. Since it is weakly quasi-coherent, it follows that N is torsion. The result then follows from the fact that a torsion object N with $N \otimes K_m \simeq 0$ for all m is contractible.

To establish the right hand horizontal equivalence in the middle row we need to show that if we have a morphism $x \rightarrow y$ where x is $wqce$, y is $wqcie$ and $x \otimes \pi \rightarrow y \otimes \pi$ is a $wqce$ -cellular equivalence for all $\pi \in \Pi$ then it is a $wqcie$ -cellular equivalence.

It is convenient to factor $x \rightarrow y$ as $x \rightarrow y^c \rightarrow y$ where y^c is wqc (i.e., the cofibrant replacement of y in $R_{wqce}\mathcal{C}_{ad}$). Now the map $y^c \rightarrow y$ is an equivalence at the vertex, an i -equivalence at the $L\Lambda\mathbb{1}$ -module, and a Π -equivalence at the nub, so it is a Π -equivalence. In other words we may assume both x and y are $wqce$, and since we are in the stable setting, we need only prove for x $wqce$ that $[k, x \otimes \pi] = 0$ for k $wqce$ implies it also holds for k $wqcie$. If k is $wqcie$ then one checks that the $wqce$ -cellularization $k^c \rightarrow k$ induces an isomorphism in $[-, x \otimes \pi]$ for $\pi \in \Pi$. This is true because $[k, x \otimes \pi] = [k \otimes \pi, x \otimes \pi]$, and $k^c \rightarrow k$ is a π -equivalence.

To establish the lower horizontal equivalences we need to show that if x is $wqce$, and y is wqc and $x \otimes \pi \rightarrow y \otimes \pi$ is a $wqce$ -cellular equivalence then it is a wqc -cellular equivalence. It is once again convenient to factor $x \rightarrow y$ as $x \rightarrow y^c \rightarrow y$ where y^c is wqc . Now the map $y^c \rightarrow y$ is an equivalence at the vertex, and a Λ -equivalence at the $\Lambda\mathbb{1}$ spot, so it is a Λ -equivalence. In other words we may assume both x and y

are wqce, and since we are in the stable setting, we need only prove for x wqce that $[k, x \otimes \lambda] = 0$ for k wqce for all $\lambda \in \Lambda$ implies it also holds for k wqc. If k is wqc then the wqc-cellularization $k^c \rightarrow k$ induces an isomorphism in $[-, x \otimes \lambda]$.

Indeed, suppose

$$\begin{aligned} k &= (M \xrightarrow{\quad} P \xleftarrow{\quad} U), \\ x &= (N \xrightarrow{\quad} Q \xleftarrow{\quad} V), \\ \lambda &= (K_m \xrightarrow{\quad} 0 \xleftarrow{\quad} L_{\mathfrak{g}}\mathbb{1}). \end{aligned}$$

Then $[k, x \otimes \lambda] = [M, N \otimes K_m] \times [U, V]$. Since $k^c \rightarrow k$ is an isomorphism at the $L_{\mathfrak{g}}\mathbb{1}$ vertex, there is nothing to check at that point. At the $\prod_m \Lambda_m \mathbb{1}$ position, we see the fibre of $M^c \rightarrow M$ is the same as the fibre at $P^c \rightarrow P$, and hence in particular it is $L_{\mathfrak{g}}X$ for some X , and $[L_{\mathfrak{g}}X, N \otimes K_m] = 0$.

This completes the proof of Theorem 4.1.

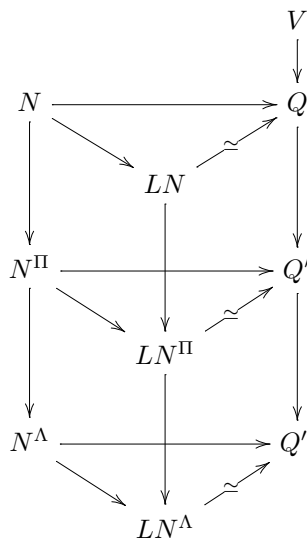
4.E. Bifibrant objects

In summary, we now have three homotopical models:

- The adelic model – $R_{\text{wqce}}\mathcal{C}_{\text{ad}}$, with bifibrant objects wqce
- The separated model – $L_{\Pi}R_{\text{wqcie}}\mathcal{C}_{\text{ad}}$, with bifibrant objects wpqcie (i.e., wqcie and the nub equivalent to a product)
- The complete model – $L_{\Lambda}R_{\text{wqc}}\mathcal{C}_{\text{ad}}$, with bifibrant objects wqc κ (i.e., wqc, with the nub complete).

To finish off Part 1 we will describe explicit constructions of moving between bifibrant replacements of one of these models to another.

To obtain a separated or complete bifibrant object from an adelic bifibrant object (i.e., a wqce-object $N \xrightarrow{\quad} Q \xleftarrow{\quad} V$) we form the diagram



where the front right faces define Q' and Q'' as homotopy pushouts and superscripts Π and Λ refer to fibrant replacements. One sees that the map $Q \rightarrow Q'$ is an idempotent equivalence since that holds for $N \rightarrow N^{\Pi}$ and this is preserved by localization.

Alternatively, given a bifibrant model in the complete or separated model, we need to upgrade the vertical from a map or an idempotent extended map and to a fully extended map, and these operate by taking homotopy pullbacks on the back face. The fact that these are homotopy pullbacks was discussed in the overall introduction.

To go from the complete to the separated model we first go to the standard model and then to the separated model.

Part 2. Algebraic models

5. Skeletons and the roadmap

In Part 2 we show how in certain algebraic situations we can replace the underlying category \mathcal{C}_{ad} of the model by a more economical skeleton. A skeleton of a model category is a Quillen equivalent subcategory \mathcal{S} in which homotopy equivalences are closer to isomorphisms, so that the homotopy relation has less work to do.

5.A. Skeletons

The simplest case is when the inclusion $i : \mathcal{S} \rightarrow \mathcal{C}$ has a *right* adjoint, that is, \mathcal{S} is *coreflective* in \mathcal{C} . The subcategory \mathcal{S} then inherits completeness and cocompleteness from \mathcal{C} . If the generating sets of cofibrations and acyclic cofibrations of \mathcal{C} lie in \mathcal{S} there is a left-lifted model structure on \mathcal{S} with the same classes of weak equivalences, cofibrations and fibrations [22] and the inclusion is a Quillen equivalence: we say \mathcal{S} is a *cellular skeleton* of \mathcal{C} .

If the inclusion $i : \mathcal{S} \rightarrow \mathcal{C}$ has a *left* adjoint, that is, \mathcal{S} is *reflective* in \mathcal{C} , then there is a little more to do. The subcategory \mathcal{S} again inherits completeness and cocompleteness from \mathcal{C} . For a simple example of this, the reader can look ahead to Section 6.B, where we describe $\prod_m(R_m\text{-mod})$ as a reflective subcategory of $\prod_m(R_m\text{-mod})$.

Under suitable hypotheses, if \mathcal{C} is cofibrantly generated we can then lift the model structure in \mathcal{C} along the right adjoint i to obtain a model structure on \mathcal{S} . If the inclusion of \mathcal{S} in \mathcal{C} is a Quillen equivalence, we say \mathcal{S} is a *local skeleton* of \mathcal{C} .

For example, if the left adjoint takes small objects to small objects, and if the unit and counit are equivalences on generators, we obtain the conclusion from the Cellularization Principle. However, in our principal examples the left adjoint does not preserve small objects and we shall see in Section 9.B that we must argue more directly.

5.B. The roadmap

We may now describe our strategy. In Part 1 we constructed three different models for \mathcal{C} with underlying category \mathcal{C}_{ad} . We are going to vary the underlying category and find models for \mathcal{C} on each one. In fact we will construct a diagram of adjoint pairs in the pattern

$$\begin{array}{ccccc}
 \mathcal{C}_{\text{ad}}^{\text{qce}} & & \mathcal{C}_{\text{sep}}^{\text{qcie}} & & \\
 \downarrow i & \uparrow \Gamma_e & \downarrow i & \uparrow \Gamma_{ie} & \\
 \mathcal{C}_{\text{ad}}^{\text{qc}} & \xrightarrow{\sigma_*} & \mathcal{C}_{\text{sep}}^{\text{qc}} & \xrightarrow{\ell_*} & \mathcal{C}_{L\kappa}^{\text{qc}} \\
 \uparrow \Gamma_{\text{qc}} & \xleftarrow{\sigma^*} & \uparrow \Gamma_{\text{qc}} & \xleftarrow{\ell^*} & \uparrow \Gamma_{\text{qc}} \\
 \mathcal{C}_{\text{ad}} & & \mathcal{C}_{\text{sep}} & & \mathcal{C}_{L\kappa}
 \end{array}$$

The categories $\mathcal{C}_{\text{ad}}, \mathcal{C}_{\text{sep}}$ and $\mathcal{C}_{L\kappa}$ at the bottom are categories of diagrams on the cospans ad, sep and $L\kappa$ to be introduced below. Above each of the three there are subcategories. In each case, the letter i indicates a coreflective inclusion with right adjoint Γ (with appropriate subscript). The functors Γ are introduced in Section 8. The horizontal adjoint pairs are induced from adjoint pairs at one vertex of the cospan by a construction to be described in Subsection 7.B. We will show that all the categories in the diagram except \mathcal{C}_{sep} and $\mathcal{C}_{L\kappa}$ inherit model structures lifted from that on \mathcal{C}_{ad} .

The diagram serves as a roadmap. The minimal skeletons of the adelic, separated and complete models are at the tops of the columns. The fact that vertical inclusions i are Quillen equivalences follows by the cellular skeleton argument above: this only requires us to show that the model structures are generated by cofibrations and acyclic cofibrations from the top category in each column. In effect we need to show an object for which the qc, qce, qcie condition holds weakly is equivalent to a subobject for which is holds strongly.

For the horizontal adjunctions, the categories decrease in size as we move right. To show that they are inclusions of local skeletons, we use corresponding generators and show that the unit and counit are weak equivalences.

5.C. *The equivalences*

Here we shall give an overview of the argument: full details are provided in Section 9.

If we start from the adelic model $R_{\text{wqce}}\mathcal{C}_{\text{ad}}$ at the bottom left and the passage to cellular skeletons in the first column shows

$$R_{\text{wqce}}\mathcal{C}_{\text{ad}} = R_{\text{we}}R_{\text{wqc}}\mathcal{C}_{\text{ad}} \simeq_Q R_{\text{we}}\mathcal{C}_{\text{ad}}^{\text{qc}} \simeq_Q \mathcal{C}_{\text{ad}}^{\text{qce}}$$

and the cellular skeleton of the adelic model is $\mathcal{C}_{\text{ad}}^{\text{qce}}$.

If we start from the separated model $L_{\Pi}R_{\text{wqc}}\mathcal{C}_{\text{ad}}$ we have the equivalences

$$L_{\Pi}R_{\text{wqc}}\mathcal{C}_{\text{ad}} = L_{\Pi}R_{\text{wie}}R_{\text{wqc}}\mathcal{C}_{\text{ad}} \stackrel{(1)}{\simeq}_Q L_{\Pi}R_{\text{wie}}\mathcal{C}_{\text{ad}}^{\text{qc}} \stackrel{(2)}{\simeq}_Q R_{\text{wie}}\mathcal{C}_{\text{sep}}^{\text{qc}} \stackrel{(3)}{\simeq}_Q \mathcal{C}_{\text{sep}}^{\text{qc}}\text{cie}$$

where (1) and (3) are cellular skeleton equivalences and (2) is a local skeleton equivalence. The skeleton of the separated model is $\mathcal{C}_{\text{sep}}^{\text{qc}}\text{cie}$.

If we start from the complete model $L_{\Lambda}R_{\text{wqc}}\mathcal{C}_{\text{ad}}$ we have the equivalences

$$L_{\Lambda}R_{\text{wqc}}\mathcal{C}_{\text{ad}} \stackrel{(1)}{\simeq}_Q L_{\Lambda}\mathcal{C}_{\text{ad}}^{\text{qc}} = L_{\Lambda}L_{\Pi}\mathcal{C}_{\text{ad}}^{\text{qc}} \stackrel{(2)}{\simeq}_Q L_{\Lambda}\mathcal{C}_{\text{sep}}^{\text{qc}} \stackrel{(3)}{\simeq}_Q \mathcal{C}_{L\kappa}^{\text{qc}}$$

where (1) is a cellular skeleton equivalence, whilst (2) and (3) are local skeleton equivalences. The skeleton of the complete model is $\mathcal{C}_{L\kappa}^{\text{qc}}$.

It remains to introduce the categories and functors appearing in the roadmap diagram, which we do in Sections 7 and 8. After this we shall show that objects with weak conditions are equivalent to ones with strong conditions. Finally we establish that the unit and counits of the horizontal adjunctions are derived equivalences on generators: this will be done in Section 9.

6. **Local skeletons of the nub**

As described above, there are two ways of moving to a smaller skeleton. One of them changes the categories in the diagram and one of them places restrictions on the objects for a fixed diagram. The change of categories in the diagram is passage to a local skeleton, and is based on changing the nub category $(\prod_m \Lambda_m \mathbb{1})\text{-mod}$ to a local skeleton. In this section we introduce the functors we use to change the nub and in Section 7 we explain how to lift this to diagrams.

6.A. Pattern

In both cases the argument proceeds as follows. We have an inclusion $i : \mathcal{N}' \rightarrow \mathcal{N}$ of a subcategory with a left adjoint F , and a left Bousfield localization L of \mathcal{N} . In our situation this gives a diagram of left Quillen functors

$$\begin{array}{ccc}
 \mathcal{N} & \xrightarrow{F} & \mathcal{N}' \\
 \downarrow & \nearrow \bar{F} & \\
 L\mathcal{N} & & \simeq_Q \mathcal{N}'
 \end{array}$$

In other words, we have a Quillen pair $F \dashv i$ relating \mathcal{N} to \mathcal{N}' , but F is also a left Quillen functor from $L\mathcal{N}$ and the same adjoint pair gives a Quillen equivalence $L\mathcal{N} \simeq_Q \mathcal{N}'$.

6.B. The separated skeleton

Let $R_m = \Lambda_m \mathbb{1}$ in a Noetherian model category \mathcal{C} . There are many modules over the product ring $\prod_m R_m$, but the ones we care most about are those which are products of modules over the individual rings R_m . Using the projective model structure, $\prod_m (R_m\text{-mod})$ is a local skeleton of $(\prod_m R_m)\text{-mod}$.

As in Subsection 4.A.2, the functor

$$\sigma : \left(\prod_m R_m\right)\text{-mod} \longrightarrow \prod_m (R_m\text{-mod})$$

defined by

$$\sigma(M)_m = e_m M := R_m \otimes_{\prod_m R_m} M$$

has right adjoint

$$\pi : (R_m\text{-mod}) \longrightarrow \left(\prod_m R_m\right)\text{-mod}$$

defined by

$$\pi(\{N_m\}) = \prod_m N_m.$$

To see that σ is the inclusion of a reflective subcategory we note

$$N_m \xrightarrow{\cong} R_m \otimes_{\prod_m R_m} \prod_m N_m$$

is an isomorphism.

Lemma 6.1. *The functor $\sigma : (\prod_m R_m)\text{-mod} \rightarrow \prod_m (R_m\text{-mod})$ is the inclusion of a reflective subcategory and $\prod_m (R_m\text{-mod})$ is a local skeleton of $L_{\Pi}(\prod_m R_m)\text{-mod}$.*

Proof. We take the projective model structure on $(\prod_m R_m)\text{-mod}$ and similarly on each of the factors of $\prod_m (R_m\text{-mod})$. We see that $\sigma \dashv \pi$ is a Quillen adjunction, and we may lift the model structure along the right adjoint.

Indeed, this same adjunction gives a Quillen pair $\bar{\sigma} : L_{\Pi}(\prod_m R_m)\text{-mod} \rightarrow \prod_m (R_m\text{-mod})$. One checks directly that this is a Quillen equivalence. \square

6.C. The L -complete skeleton

Unlike the separated skeleton, the L -complete skeleton will only exist under some hypothesis on $\Lambda_{\mathfrak{m}}\mathbb{1}$. As such, we now restrict to the situation that $R_{\mathfrak{m}}$ is a graded commutative Noetherian local ring with maximal ideal \mathfrak{m} . We would like to replace $R_{\mathfrak{m}}\text{-mod}$ by a skeleton which is complete in some sense. The first thought is to replace $R_{\mathfrak{m}}\text{-mod}$ by its Bousfield completion $\Lambda_{\mathfrak{m}}(R_{\mathfrak{m}}\text{-mod})$ however this is not helpful for the skeleton since the underlying subcategory remains the same.

It is natural to seek an abelian model. One thinks first of the category of \mathfrak{m} -adically complete modules, but since \mathfrak{m} -adic completion is not exact, the category is not well behaved. Nonetheless, Pol and Williamson have shown the completion is the derived category of a convenient abelian category. Writing $L_0^{\mathfrak{m}}$ for the 0th left derived functor $L_0^{\mathfrak{m}}$ of \mathfrak{m} -adic completion, we take the $L_0^{\mathfrak{m}}$ -complete modules M (those for which the natural transformation $M \rightarrow L_0^{\mathfrak{m}}M$ is an isomorphism [17]). The category $L_0^{\mathfrak{m}}\text{-}R_{\mathfrak{m}}\text{-mod}$ of differential graded $L_0^{\mathfrak{m}}$ -complete modules is a very well behaved reflective subcategory of $R_{\mathfrak{m}}\text{-mod}$ and gives a local skeleton. The following result is stated in [25, Appendix A], [7, Appendix A]; both references have a blanket assumption that their rings are regular and local, but these assumptions are unnecessary for the given proofs of the following statement.

Lemma 6.2.

- The category $L_0^{\mathfrak{m}}\text{-}R_{\mathfrak{m}}\text{-mod}$ is abelian, and the inclusion functor $i: L_0^{\mathfrak{m}}\text{-}R_{\mathfrak{m}}\text{-mod} \hookrightarrow R_{\mathfrak{m}}\text{-mod}$ is exact.
- The inclusion functor has left adjoint $L_0^{\mathfrak{m}}$.
- The category $L_0^{\mathfrak{m}}\text{-}R_{\mathfrak{m}}\text{-mod}$ is complete and cocomplete.

On the basis of this we have an abelian local skeleton of $\Lambda_{\mathfrak{m}}(R_{\mathfrak{m}}\text{-mod})$.

Proposition 6.3 ([28, 6.10]). *There is a cofibrantly generated monoidal model structure on $L_0^{\mathfrak{m}}\text{-}R_{\mathfrak{m}}\text{-mod}$ where a map is a weak equivalence or fibration if it is when viewed in $R_{\mathfrak{m}}\text{-mod}$: the model structure is right-lifted from that in $R_{\mathfrak{m}}\text{-mod}$.*

Moreover, there is a symmetric monoidal Quillen equivalence

$$L_0^{\mathfrak{m}} : \Lambda_{\mathfrak{m}}(R_{\mathfrak{m}}\text{-mod}) \rightleftarrows L_0^{\mathfrak{m}}\text{-}R_{\mathfrak{m}}\text{-mod} : i.$$

We note that $L_0^{\mathfrak{m}}$ does not preserve sequential colimits, so this requires a direct proof rather than using the general right-lifting criterion.

7. The adelic, separated and L -complete categories

Having described two left adjoints at the nub category, we need to explain how to this may be extended to the whole cospan. In Section 9.B we describe how to equip these categories with relevant model structures.

7.A. Quasi-coherent diagram categories

We will name and display the left adjoints, with notation for the right adjoints following from it, so that if $f: \mathcal{C} \rightarrow \mathcal{C}'$ is a left adjoint the right adjoint will be called f^r .

We start from a cospan of left adjoints, but since we focus on quasi-coherent objects, the categories do not play equal roles, and the notation reflects this. The cospan of left adjoints is the diagram $\alpha = (\mathcal{N} \xrightarrow{v} \mathcal{Q} \xleftarrow{h} \mathcal{V})$, where h and v stand for ‘horizontal’ and ‘vertical’ and we have the corresponding category

$$\mathcal{C}_\alpha = \mathcal{C}_\alpha(\mathcal{N}, h) = \Gamma \left(\begin{array}{ccc} & & \mathcal{V} \\ & & \downarrow v \\ \mathcal{N} & \xrightarrow{h} & \mathcal{Q} \end{array} \right)$$

of diagrams. The vertical functor $v : \mathcal{V} \rightarrow \mathcal{Q}$ will remain constant throughout the discussion so is not recorded in the notation. Given a left adjoint $F : \mathcal{N} \rightarrow \mathcal{N}'$ we may form a new cospan

$$F_*\alpha = (\mathcal{N}' \xrightarrow{F^r} \mathcal{N} \xrightarrow{h} \mathcal{Q} \leftarrow \mathcal{V})$$

where the horizontal h has been replaced by the composite hF^r . We note that the horizontal of $F_*\alpha$ is usually not a left adjoint. Nonetheless, we can consider the category of diagrams

$$\mathcal{C}_{F_*\alpha} = \mathcal{C}_{F_*\alpha}(\mathcal{N}', hF^r) = \Gamma \left(\begin{array}{ccc} & & \mathcal{V} \\ & & \downarrow v \\ \mathcal{N}' & \xrightarrow{F^r} \mathcal{N} \xrightarrow{h} & \mathcal{Q} \end{array} \right)$$

7.B. Quasi-coherent pushout along a left adjoint

Our main focus is the category

$$\mathcal{C}_\alpha^{qc} = \mathcal{C}_\alpha^{qc}(\mathcal{N}, h) = \Gamma \left(\begin{array}{ccc} & & \mathcal{V} \\ & & \downarrow v \\ \mathcal{N} & \xrightarrow{\bullet} \xrightarrow{h} & \mathcal{Q} \end{array} \right)$$

of qc objects. Since the base change along h is an isomorphism, we may think of a qc object

$$Y = \left(\begin{array}{ccc} & & V \\ & & \vdots \\ & & v \\ N & \cdots \bullet \xrightarrow{\quad} & Q \end{array} \right)$$

as the nub N together with the additional structure of a vertex V and a map $i : v(V) \rightarrow h(N)$ in the splicing category \mathcal{Q} .

We describe how a left adjoint $F : \mathcal{N} \rightarrow \mathcal{N}'$ induces comparisons between qc diagrams with different nubs.

Lemma 7.1. *A left adjoint $F : \mathcal{N} \rightarrow \mathcal{N}'$ induces a left adjoint $F_* : \mathcal{C}_\alpha^{qc} \rightarrow \mathcal{C}_{F_*\alpha}^{qc}$. If F is the inclusion of a reflective subcategory, so is F_* .*

Proof. The left adjoint F_* is defined by

$$F_*Y = \left(\begin{array}{c} V \\ \vdots \downarrow v \\ hN \\ \downarrow h\eta \\ FN \cdots \bullet \rightarrow hF^r FN \end{array} \right)$$

where $\eta : N \rightarrow F^r FN$ is the unit of the adjunction. The right adjoint $F^* = F^*$ is defined by

$$F^*Y' = \left(\begin{array}{c} V' \\ \vdots \downarrow v \\ F^r N' \cdots \bullet \rightarrow hF^r N' \end{array} \right)$$

The unit $\hat{\eta} : Y \rightarrow F^*F_*Y$ of the adjunction is given by the diagram

$$\begin{array}{ccccc} & & V & \xrightarrow{=} & V \\ & & \vdots \downarrow & & \vdots \downarrow \\ N & \cdots \rightarrow & hN & \xrightarrow{=} & hN \\ \downarrow & & \searrow & & \downarrow \\ F^r FN & \cdots \rightarrow & & & hF^r FN \end{array}$$

The counit $\hat{\epsilon} : F_*F^*Y' \rightarrow Y'$ of the adjunction is given by the diagram

$$\begin{array}{ccccc} & & V' & & \\ & & \searrow = & & \\ & & & & V' \\ & & \vdots \downarrow & & \vdots \downarrow \\ & & hF^r N' & & \\ & & \downarrow & & \\ FF^r N' & \cdots \rightarrow & hF^r FF^r N' & & \\ \searrow & & \searrow & & \downarrow \\ N' & \cdots \rightarrow & & & hF^r N' \end{array}$$

where commutativity of the triangle is given by the triangular identity of the original adjunction. The triangular identities for $\hat{\eta}$ and $\hat{\epsilon}$ follow directly from the triangular identities of η and ϵ at the nub. \square

7.C. The separated cospan

The starting point is the adelic diagram $\alpha = \text{ad}$, so that in $C_{\text{ad}}^{\text{qc}}$, the nub is a module over the product ring $\prod_{\mathfrak{m}} R_{\mathfrak{m}}$ where $R_{\mathfrak{m}} = \Lambda_{\mathfrak{m}} \mathbb{1}$. In the expanded notation,

$$\mathcal{C}_{\text{ad}}^{\text{qc}} = \mathcal{C}_{\text{ad}}^{\text{qc}}((\prod_{\mathfrak{m}} R_{\mathfrak{m}})\text{-mod}, L_{\mathfrak{g}}).$$

We now use the left adjoint

$$\sigma : (\prod_{\mathfrak{m}} R_{\mathfrak{m}})\text{-mod} \longrightarrow \prod_{\mathfrak{m}} (R_{\mathfrak{m}}\text{-mod})$$

from Subsection 6.B.

Definition 7.2. The category \mathcal{C}_{sep} is the category of sections of the *separated cospan*:

$$\mathcal{C}_{\text{sep}} = \mathcal{C}_{\sigma_* \text{ad}} = \Gamma \left(\begin{array}{ccc} & & (L_{\mathfrak{g}}\mathbb{1})\text{-mod}_{\mathcal{C}} \\ & & \downarrow j_* \\ \prod_{\mathfrak{m}} [\Lambda_{\mathfrak{m}}\mathbb{1}\text{-mod}_{\mathcal{C}}] & \xrightarrow{L_{\mathfrak{g}}\pi} & (L_{\mathfrak{g}}\prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}}\mathbb{1})\text{-mod}_{\mathcal{C}} \end{array} \right)$$

Unpacking the definition, the horizontal functor $L_{\mathfrak{g}}\pi$ takes a collection of modules $\{M_{\mathfrak{m}}\}_{\mathfrak{m}}$ to the module $L_{\mathfrak{g}}\prod_{\mathfrak{m}} M_{\mathfrak{m}}$, so that an object of \mathcal{C}_{sep} is a diagram of the form

$$X = \left[\begin{array}{ccc} & & U \\ & & \vdots \\ & & Y \\ \{M_{\mathfrak{m}}\}_{\mathfrak{m}} & \cdots \cdots \cdots > & P \end{array} \right]$$

Its base change maps are $j_*U \rightarrow P$ and $L_{\mathfrak{g}}\prod_{\mathfrak{m}} M_{\mathfrak{m}} \rightarrow P$.

Our model is built from the qc objects $\mathcal{C}_{\text{sep}}^{\text{qc}}$ in \mathcal{C}_{sep} . The construction of Section 7.B shows that the $\sigma \dashv \pi$ adjunction induces an adjunction with left adjoint

$$\sigma_* : \mathcal{C}_{\text{ad}}^{\text{qc}} = \mathcal{C}_{\text{ad}}^{\text{qc}}(L_{\mathfrak{g}}) \longrightarrow \mathcal{C}_{\text{sep}}^{\text{qc}}(L_{\mathfrak{g}}\pi) = \mathcal{C}_{\text{sep}}^{\text{qc}}.$$

7.D. The L -complete cospan

We use the L -complete local skeleton of Subsection 6.C and the constructions of Section 7.B to replace the nub $\prod_{\mathfrak{m}} (\Lambda_{\mathfrak{m}}\mathbb{1}\text{-mod})$ by a complete category, but we need to restrict the categories to the appropriate context.

We restrict attention to the special case where individual categories $\Lambda_{\mathfrak{m}}(\mathcal{C}_{\mathfrak{m}})$ are algebraic in a strong sense.

Definition 7.3. We say that \mathcal{C} is *formally algebraic* if, for each closed point \mathfrak{m} , $\Lambda_{\mathfrak{m}}\mathbb{1}$ is formal in the sense that we can find a Noetherian graded commutative local ring $(R_{\mathfrak{m}}, \mathfrak{m})$ and a Quillen equivalence $\Lambda_{\mathfrak{m}}\mathbb{1}\text{-mod}_{\mathcal{C}} \simeq R_{\mathfrak{m}}\text{-mod}$.

Note that we have abused notation by using \mathfrak{m} to refer to both the Balmer prime $\mathfrak{m} \in \text{Spc}^{\omega}(\mathcal{C})$ on the left, and the algebraic prime $\mathfrak{m} \in \text{Spec}(R_{\mathfrak{m}})$ on the right.

Example 7.4.

- (i) The most obvious example is if $\mathcal{C} = R\text{-mod}$ of (chain complexes of) R -modules for a 1-dimensional commutative Noetherian domain, so that all the completions come from a single ring: in this case $(R_{\mathfrak{m}}, \mathfrak{m}) = (R_{\mathfrak{m}}^{\wedge}, \mathfrak{m})$. The results are valuable even in this case.

(ii) We will also consider the example when \mathcal{C} consists of (chain complexes of) quasi-coherent sheaves over a curve C . In this case \mathfrak{m} runs through closed points and $\Lambda_{\mathfrak{m}}\mathcal{O}_C \simeq (\mathcal{O}_C)_{\mathfrak{m}}^{\wedge}$ is a skyscraper sheaf, and we may take $R_{\mathfrak{m}} = \Gamma(C; (\mathcal{O}_C)_{\mathfrak{m}}^{\wedge})$.

As in Subsection 6.C, we let $L_0^{\mathfrak{m}}\text{-}R_{\mathfrak{m}}\text{-mod}$ denote the category of differential graded $L_0^{\mathfrak{m}}$ -complete modules. We consider the product

$$\ell : \prod_{\mathfrak{m}} (R_{\mathfrak{m}}\text{-mod}) \longrightarrow \prod_{\mathfrak{m}} (L_0^{\mathfrak{m}} - R_{\mathfrak{m}}\text{-mod})$$

of the individual left adjoints

$$L_0^{\mathfrak{m}} : R_{\mathfrak{m}}\text{-mod} \longrightarrow L_0^{\mathfrak{m}} - \Lambda_{\mathfrak{m}}\mathbb{1}\text{-mod}.$$

We may now define a skeleton of the complete model.

Definition 7.5. Suppose \mathcal{C} is formally algebraic. We define the L -complete cospan $\mathcal{C}_{L\kappa ad}$ and the associated category of diagrams

$$\mathcal{C}_{L\kappa} = \Gamma \left(\begin{array}{ccc} & & (L_{\mathfrak{g}}\mathbb{1})\text{-mod} \\ & & \downarrow j_* \\ \prod_{\mathfrak{m}} [L_0^{\mathfrak{m}}\text{-}R_{\mathfrak{m}}\text{-mod}] & \xrightarrow{L_{\mathfrak{g}}\ell} & (L_{\mathfrak{g}} \prod_{\mathfrak{m}} \Lambda_{\mathfrak{m}} R_{\mathfrak{m}})\text{-mod} \end{array} \right)$$

Our model is built from the qc objects $\mathcal{C}_{L\kappa}^{\text{qc}}$ in $\mathcal{C}_{L\kappa}$. The construction of Section 7.B shows that the left adjoint ℓ induces a left adjoint

$$\ell_* : \mathcal{C}_{\text{sep}}^{\text{qc}} = \mathcal{C}_{\text{sep}}^{\text{qc}}(L_{\mathfrak{g}}\pi) \longrightarrow \mathcal{C}_{L\kappa}^{\text{qc}}(L_{\mathfrak{g}}\pi\ell) = \mathcal{C}_{L\kappa}^{\text{qc}}.$$

8. The cellular skeletons

We have explained that given a coreflective inclusion $i : \mathcal{S} \longrightarrow \mathcal{C}$ of a subcategory, then if \mathcal{C} has a cofibrantly generated model structure with generators in \mathcal{S} then by [22] the inclusion is an equivalence. To fill in the vertical equivalences in the roadmap of Section 5 it remains only to construct the right adjoints and observe the generators come from the smaller subcategories.

8.A. Quasi-coherification

The quasi-coherification $\Gamma_{qc}X$ of an object

$$X = \left[\begin{array}{ccc} & & V \\ & & \vdots \\ & & Q \\ N & \cdots \longrightarrow & Q \end{array} \right]$$

of \mathcal{C}_{ad} is defined by

$$\Gamma_{\text{qc}} \left[\begin{array}{ccc} & V & \\ & \downarrow & \\ N & \dashrightarrow & Q \end{array} \right] = \left[\begin{array}{ccc} & V' & \longrightarrow & V \\ & \downarrow & \lrcorner & \downarrow \\ N & \dashrightarrow & L_{\mathfrak{g}}N & \longrightarrow & Q \end{array} \right] = \left[\begin{array}{ccc} & V' & \\ & \downarrow & \\ N & \dashrightarrow & L_{\mathfrak{g}}N \end{array} \right]$$

As before, a solid bullet denotes an isomorphism after base change. In more detail, the dotted arrow from N to Q means we have a map $L_{\mathfrak{g}}N \rightarrow Q$ and we obtain V' by pulling back along it. The object Γ_{qc} is highlighted in red.

The two-coloured display in the definition shows the counit $i\Gamma_{\text{qc}}X \rightarrow X$. It is also obvious that if X is qc then $\Gamma_{\text{qc}}iX = X$. The triangular identities are immediate, so that we have an adjunction

$$\mathcal{C}_{\text{ad}}^{\text{qc}} \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{\Gamma_{\text{qc}}} \end{array} \mathcal{C}_{\text{ad}}$$

The definition of Γ_{qc} on \mathcal{C}_{sep} and $\mathcal{C}_{L\kappa}$ is exactly similar, and it is easy to see they are also right adjoint to the inclusion. This gives the bottom half of the road map in Subsection 5.B.

8.B. Extensification

The definition of Γ_e is very much like that of Γ_{qc} :

$$\Gamma_e \left[\begin{array}{ccc} & V & \\ & \downarrow & \\ N & \dashrightarrow & Q \end{array} \right] = \left[\begin{array}{ccc} & V & \\ & \bullet & \\ & \downarrow & \\ N' & \dashrightarrow & j_*V \\ \downarrow & & \downarrow \\ N & \dashrightarrow & Q \end{array} \right] = \left[\begin{array}{ccc} & V & \\ & \bullet & \\ & \downarrow & \\ N' & \dashrightarrow & j_*V \end{array} \right]$$

Once again, N' is defined as a pullback and $\Gamma_e X$ is highlighted in red. Once again, the two coloured display in the definition shows the counit $i\Gamma_e X \rightarrow X$. It is also obvious that if X is extended then $\Gamma_e iX = X$. The triangular identities are immediate, so that we have an adjunction

$$\mathcal{C}_{\text{ad}}^e \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{\Gamma_{\text{qc}}} \end{array} \mathcal{C}_{\text{ad}}$$

8.C. The quasi-coherification extensification functor

In order to construct an adjoint to the inclusion $\mathcal{C}_{\text{ad}}^{\text{qce}} \rightarrow \mathcal{C}_{\text{ad}}^{\text{qc}}$ we need an additional condition.

The Bousfield localization $L_{\mathfrak{g}}$ is smashing in the sense that $L_{\mathfrak{g}}X \simeq L_{\mathfrak{g}}\mathbb{1} \otimes X$, and it follows that multiplication gives a weak equivalence $L_{\mathfrak{g}}\mathbb{1} \otimes L_{\mathfrak{g}}\mathbb{1} \simeq L_{\mathfrak{g}}\mathbb{1}$. When forming algebraic models, we may obtain a leaner model by requiring this to hold more strictly.

Definition 8.1. We say that a localization functor L is *strictly smashing* if the multiplication map $L\mathbb{1} \otimes L\mathbb{1} \rightarrow L\mathbb{1}$ is an isomorphism and $L\mathbb{1}$ is flat.

Example 8.2.

- (i) If \mathcal{C} is the category of (chain complexes of) R -modules for a commutative ring R and L is localization to invert a multiplicatively closed set, then L is strictly smashing.

(ii) If \mathcal{C} is the category of (chain complexes of) quasi-coherent sheaves on a curve C the generic localization $L_{\mathfrak{g}}$ is passage to stalks over the generic point, and this is also strictly smashing.

Definition 8.3. Let \mathcal{C} be a 1-dimensional Noetherian model category. We will say that \mathcal{C} has *strict generization* if $L_{\mathfrak{g}}$ is strictly smashing.

All the examples we have in mind are based on categories of differential graded modules, so they are simplicial and have strict generization essentially as in Example 8.2.

Lemma 8.4. *If we have strict generization, then the functor Γ_e preserves qc-modules. Hence $\Gamma_{\text{qce}} = \Gamma_e \Gamma_{\text{qc}}$ is the right adjoint to the inclusion $i : \mathcal{C}_{\text{ad}}^{\text{qce}} \rightarrow \mathcal{C}_{\text{ad}}$.*

Proof. It is clear that Γ_{qc} lands in the category where the horizontal base change map is an isomorphism. As such, it is enough to assume that we have $X \in \mathcal{C}_{\text{ad}}^{\text{qc}}$ and that applying Γ_e preserves the property of being qc. We set

$$X = \left[\begin{array}{ccc} & & V \\ & & \vdots \\ & & \downarrow \\ N \cdots \bullet & \xrightarrow{\quad} & L_{\mathfrak{g}}N \end{array} \right]$$

Then

$$\Gamma_{\text{qce}}X = \Gamma_e X = \left[\begin{array}{ccc} & & V \\ & & \vdots \\ & & \downarrow \\ N' \cdots \bullet & \xrightarrow{\quad} & j_*V \\ \downarrow & & \downarrow \\ N \cdots \bullet & \xrightarrow{\quad} & L_{\mathfrak{g}}N \end{array} \right]$$

We need to show that $L_{\mathfrak{g}}N' \cong j_*V$, but this follows from the fact that $L_{\mathfrak{g}}$ is strictly smashing: we apply $L_{\mathfrak{g}}$ to the diagram to view it in the category of $L_{\mathfrak{g}}\mathbb{1}$ -modules and note that pullbacks preserve isomorphisms. \square

8.D. *The idempotent extendification functor*

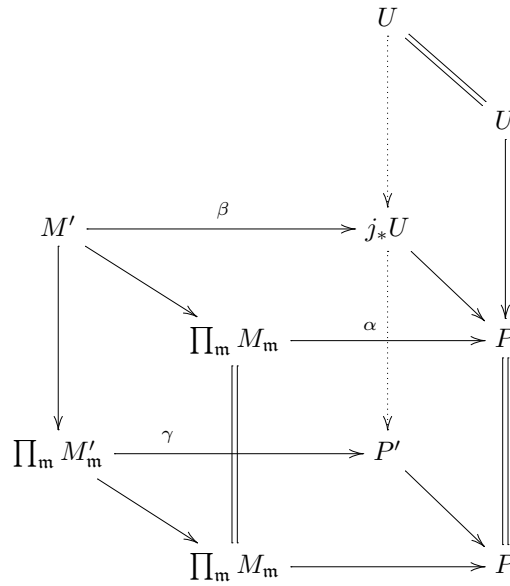
The remaining torsion functor is a little more complicated.

Lemma 8.5. *The inclusion $\mathcal{C}_{\text{sep}}^{\text{ie}} \rightarrow \mathcal{C}_{\text{sep}}$ has a right adjoint $\Gamma_{\text{ie}} : \mathcal{C}_{\text{sep}} \rightarrow \mathcal{C}_{\text{sep}}^{\text{ie}}$, and if we have strict generization, Γ_{ie} restricts to a functor $\Gamma_{\text{ie}} : \mathcal{C}_{\text{sep}}^{\text{qc}} \rightarrow \mathcal{C}_{\text{sep}}^{\text{qcie}}$ right adjoint to the inclusion $\mathcal{C}_{\text{sep}}^{\text{qcie}} \rightarrow \mathcal{C}_{\text{sep}}^{\text{qc}}$.*

Proof. We suppose given an object X as above and define

$$\Gamma_{\text{ie}}X = \left[\begin{array}{ccc} & & U \\ & & \vdots \\ & & \downarrow \\ \{M'_m\}_m \cdots \bullet & \xrightarrow{\quad} & j_*U \end{array} \right]$$

where the factors M'_m are defined by the diagram



where M' is a pullback of j_*U and $\prod_m M_m$ and $M'_m = e_m M'$, and P' is defined as a pushout. The pullback and pushout are taken in $\Lambda_m \mathbf{1}$ -modules, with P' local because $L_{\mathfrak{g}}$ preserves pushouts. Because the back left hand vertical is ie, so is the lower back left hand vertical, so $\Gamma_{ie} X$ is indeed idempotent extended. It is clear that Γ_{ie} is the identity on idempotent extended objects, so the unit of the adjunction is the identity. The counit $i\Gamma_{ie} X \rightarrow X$ is displayed in the diagram above, and the triangular identities are easily checked.

Furthermore, if the base change of α is an isomorphism, it follows that the base change of β is an isomorphism by and hence that the base change of γ is an isomorphism. \square

9. The skeletal Quillen equivalences

Referring to the categories in the Roadmap (Subsection 5.B), we first need to consider the existence of model structures, and then establish the Quillen equivalences.

9.A. Cellular skeletons of the adelic model

By Lemma 8.4, \mathcal{C}_{ad}^{qce} is a coreflective subcategory of the cofibrantly generated model category \mathcal{C}_{ad} . To see that it admits a left-lifted model structure it suffices to show the generating cofibrations and acyclic cofibrations are also qce objects [22].

Lemma 9.1. *For a weakly qce object X , the counit $\Gamma_{qce} X \xrightarrow{\cong} X$ is an equivalence in $R_{wqce} \mathcal{C}_{ad}$, so cellularization with respect to wqce coincides with that for qce.*

For a weakly qc object X , the counit $\Gamma_{qc} X \xrightarrow{\cong} X$ is an equivalence in $R_{wqc} \mathcal{C}_{ad}$, so cellularization with respect to wqc coincides with that for qc.

Proof. We will provide a chain of weak equivalences $\Gamma_{qce} X \xrightarrow{\cong} \Gamma_{qc} X \xrightarrow{\cong} X$. We start by fixing our object of interest

$$X = \left[\begin{array}{c} V \\ \vdots \\ \circ \\ \downarrow \\ N \cdots \circ \rightarrow Q \end{array} \right] \in \mathcal{C}_{ad}^{wqce}$$

First we show that there is a weak equivalence

$$\Gamma_{\text{qc}}X = \left[\begin{array}{ccc} & V' & \longrightarrow V \\ & \vdots & \downarrow \\ & \circ & \downarrow \\ N \cdots \bullet \rightarrow & L_{\mathfrak{g}}N & \xrightarrow{\simeq} Q \end{array} \right] \xrightarrow{\simeq} X.$$

There is no change at the N vertex, and by assumption of X being a weak qce-diagram the map $L_{\mathfrak{g}}N \rightarrow Q$ is a weak equivalence. Therefore we are left to show that the map $V' \rightarrow V$ is a weak equivalence. However this is the base change of the weak equivalence $L_{\mathfrak{g}}N \xrightarrow{\simeq} Q$ and the model category \mathcal{C} is assumed to be proper.

Now we show that there is a weak equivalence $\Gamma_{\text{qce}}X \rightarrow \Gamma_{\text{qc}}X$. The object $\Gamma_{\text{qce}}X$ is as follows

$$\left[\begin{array}{ccc} & V' & \\ & \vdots & \\ & \circ & \\ N' \cdots \bullet \rightarrow & j_*V' & \\ \vdots & \vdots & \\ N \cdots \bullet \rightarrow & L_{\mathfrak{g}}N & \end{array} \right]$$

There is no change at the V' vertex, so we need to check at the other two points. We must show the maps $N' \rightarrow N$ and $j_*V' \rightarrow L_{\mathfrak{g}}N$ are weak equivalences. As the $\Gamma_{\text{qc}}(X)$ is weakly extended, we have that $j_*V' \rightarrow L_{\mathfrak{g}}N$ is a weak equivalence. It then follows that $N' \rightarrow N$ is also a weak equivalence as we are pulling back along an isomorphism. \square

Corollary 9.2 (Adelic cellular skeleton theorem). *Let \mathcal{C} be a 1-dimensional Noetherian model category admitting strict generization. Then inclusions give Quillen equivalences*

$$\mathcal{C} \simeq_Q R_{\text{wqce}}\mathcal{C}_{\text{ad}} \simeq_Q R_{\text{we}}\mathcal{C}_{\text{ad}}^{\text{qc}} \simeq_Q \mathcal{C}_{\text{ad}}^{\text{qce}}.$$

We refer to $\mathcal{C}_{\text{ad}}^{\text{qce}}$ as the cellular skeleton of the adelic model.

Proof. The adelic model is Theorem 3.2. By Subsection 8.A Lemma 8.4, $\mathcal{C}_{\text{ad}}^{\text{qc}}$ and $\mathcal{C}_{\text{ad}}^{\text{qce}}$ are coreflective subcategories of \mathcal{C}_{ad} . By Lemma 9.1 $\mathcal{C}_{\text{ad}}^{\text{qc}}$ and $\mathcal{C}_{\text{ad}}^{\text{qce}}$ have model structures left-lifted from \mathcal{C}_{ad} . By [22] these are cellular skeletons. \square

9.B. Model structures on qc-diagram categories

We consider model categories in the context of the types of categories appearing in Subsections 7.A and 7.B. Suppose $\mathcal{N}, \mathcal{V}, \mathcal{Q}$ are cofibrantly generated model categories. We give the diagram category \mathcal{C}_{α} the diagram-injective model structure, cellularize at R_{wqc} , and give $\mathcal{C}_{\alpha}^{\text{qc}}$ the left-lifted model structure.

Since hF^r is not a left adjoint, it is not clear that the entire category of sections $\mathcal{C}_{F^*\alpha}$ admits a diagram-injective model structure. The main point of this subsection is to observe that the subcategory $\mathcal{C}_{F^*\alpha}^{\text{qc}}$ of quasi-coherent objects does admit a model structure right-lifted along F^* from $\mathcal{C}_{\alpha}^{\text{qc}}$.

Lemma 9.3. *If \mathcal{N}' has cylinders, admits a model structure right-lifted along F^r from \mathcal{N} , and the functor v is faithfully flat, then $\mathcal{C}_{F^*\alpha}^{\text{qc}}$ admits a model structure right-lifted along F^* from $\mathcal{C}_{\alpha}^{\text{qc}}$. In this model structure cofibrant objects have cofibrant nubs and fibrant objects have fibrant nubs.*

Proof. For an object N of \mathcal{N} we consider the cospan $a(N) = (N \rightarrow hN \leftarrow 0)$. The functor a is left adjoint to evaluation:

$$\text{Hom}_{\mathcal{C}_\alpha^{qc}}(a(N), X) = \text{Hom}_{\mathcal{N}}(N, N_X).$$

For an object U of \mathcal{V} we consider the cospan, $b(U) = (0 \rightarrow 0 \leftarrow U)$, also left adjoint

$$\text{Hom}_{\mathcal{C}_\alpha^{qc}}(b(U), X) = \text{Hom}_{\mathcal{V}}(U, fV_X),$$

but now to the vertex-fibre: fV_X is the subobject of V_X mapping to $\ker(vV_X \rightarrow hN_X)$.

Suppose that $I_{\mathcal{N}}$ and $J_{\mathcal{N}}$ are the sets of generating cofibrations and acyclic cofibrations for \mathcal{N} and similarly for \mathcal{V} . Take $I = \{a(N), b(U) \mid N \in I_{\mathcal{N}}, U \in I_{\mathcal{V}}\}$, and similarly for J . We then see that $I - inj$ consists of morphisms with nub component in $I_{\mathcal{N}} - inj$ and vertex components with fV_X in $I_{\mathcal{V}} - inj$. The adjunctions mean that maps out of maps in I or J are detected either in the nub or in the vertex-fibre.

Now consider the requirements of [24, 2.1.19]. Conditions 1, 2, and 3 are obvious. Condition 4 is also obvious by the adjunctions since these conditions also hold at vertex and nub. Similarly for Condition 6. For Condition 5, $I - inj \subseteq J - inj$ is clear, and it remains to comment on $I - inj \subseteq \mathcal{W}$. Using the objects $a(N)$ we see that a map in $I - inj$ is an equivalence at the nub. From the strict smashing condition it follows that it is an equivalence also at the adelic vertex. Finally, using the objects $b(U)$ we see that it is an equivalence at the vertex-fibre. It remains to deduce that it is an equivalence at the vertex. Since \mathcal{N}' has cylinders, this can be done with acyclics. We have a pullback square

$$\begin{array}{ccc} fV & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ vV & \longrightarrow & hN \end{array}$$

Now we know that hN and fV are acyclic, so it follows that hV is acyclic. Since v is faithfully flat, it follows that V is acyclic as required. \square

Finally, we observe that with this model structure, if the left adjoint $F : \mathcal{N} \rightarrow \mathcal{N}'$ is a Quillen equivalence then the left adjoint $F_* : \mathcal{C}_\alpha^{qc} \rightarrow \mathcal{C}_{F_*\alpha}^{qc}$ is also a Quillen equivalence.

Suppose X is a cofibrant object of \mathcal{C}_α and Y is a fibrant object of $\mathcal{C}_{F_*\alpha}$ we must show that $F_*X \rightarrow Y$ is a weak equivalence if and only if its adjunct $X \rightarrow F^*Y$ is a weak equivalence. However F_* and F^* do not change the vertex, and at the nub they are $FN \rightarrow N'$ and $N \rightarrow F^rN'$: since a cofibrant object of \mathcal{C}_α^{qc} has cofibrant nub and a fibrant object in $\mathcal{C}_{F_*\alpha}^{qc}$ has fibrant nub, the result follows.

9.C. The skeleton of the separated model

We have shown in Subsection 8.A and Lemma 8.5 that

$$\mathcal{C}_{\text{sep}} \rightarrow \mathcal{C}_{\text{sep}}^{qc} \rightarrow \mathcal{C}_{\text{sep}}^{qcie}$$

are coreflective inclusions.

Lemma 9.4.

- (i) For a wqcie object X , the counit $\Gamma_{\text{qcie}} X \xrightarrow{\cong} X$ is an equivalence in $R_{\text{wqcie}}\mathcal{C}_{\text{sep}}$, so cellularization with respect to wqcie coincides with that for qcie.

- (ii) For a *wie qc*-object X , the counit $\Gamma_{ie}X \xrightarrow{\cong} X$ is an equivalence in $R_{wie}\mathcal{C}_{sep}^{qc}$, so cellularization with respect to *wie* coincides with that for *ie*.
- (iii) For a *weakly qc* object X , the counit $\Gamma_{qc}X \xrightarrow{\cong} X$ is an equivalence in $R_{wqc}\mathcal{C}_{sep}$, so cellularization with respect to *wqc* coincides with that for *qc*.

Proof. The proof of Part (iii) is exactly like the corresponding part in the adelic case.

For Part (i) we consider the last diagram of Lemma 8.5, and repeatedly use left and right properness. The notable point is that the map $M' \rightarrow \prod_m M_m$ is only an idempotent weak equivalence, but of course this implies that $\prod_m M'_m \rightarrow \prod_m M_m$ is an actual weak equivalence.

The argument for Part (ii) is similar. \square

Theorem 9.5 (*Separated skeleton theorem*). *Let \mathcal{C} be a 1-dimensional Noetherian model category admitting strict generization, and suppose v is faithfully flat. Then there are Quillen equivalences*

$$\mathcal{C} \simeq_Q L_{\Pi}R_{wqc}C_{ad} = L_{\Pi}R_{wie}R_{wqc}C_{ad} \simeq_Q L_{\Pi}R_{wie}C_{ad}^{qc} \simeq_Q R_{wie}C_{sep}^{qc} \simeq_Q C_{sep}^{qcie}$$

and hence the skeleton of the separated model C_{sep}^{qcie} is Quillen equivalent to \mathcal{C} .

Proof. By Theorem 4.1, \mathcal{C} is equivalent to the separated model $L_{\Pi}R_{wqc}C_{ad}$. The equivalence $L_{\Pi}R_{wie}R_{wqc}C_{ad} \simeq L_{\Pi}R_{wie}C_{ad}^{qc}$ follows from the fact that C_{ad}^{qc} is a cellular skeleton of $R_{wqc}C_{ad}$ (Corollary 9.2). The equivalence $R_{wie}C_{sep}^{qc} \simeq C_{sep}^{qcie}$ follows from Lemma 9.4 since the objects of \mathcal{G}_{ad} are in the image of the reflective inclusion σ^* .

The equivalence $L_{\Pi}R_{wie}C_{ad}^{qc} \simeq L_{\Pi}R_{wie}C_{sep}^{qc}$ is based on the adjoints from Subsection 7.C. Subsection 9.B shows that C_{sep}^{qc} admits a model structure right-lifted from C_{ad}^{qc} . Accordingly, the $\sigma_* \dashv \sigma^*$ adjunction is Quillen for the model structures on C_{ad}^{qc} and C_{sep}^{qc} . Cellularizing both sides, it remains Quillen for the model structures on $R_{we}C_{ad}^{qc}$ and $R_{wie}C_{sep}^{qc}$. Finally, it remains a Quillen adjunction relating $L_{\Pi}R_{we}C_{ad}^{qc}$ and $R_{wie}C_{sep}^{qc}$, and this adjunction is a Quillen equivalence from the definition. \square

Remark 9.6. We may consider the following ‘super-separated’ diagram

$$C_{ssep} = \Gamma \left(\begin{array}{ccc} & & (L_{\mathfrak{g}}\mathbb{1})\text{-mod}_{\mathcal{C}} \\ & & \downarrow \{e_m j_*\} \\ \prod_m [\Lambda_m \mathbb{1}\text{-mod}_{\mathcal{C}}] & \xrightarrow{\prod_m L_{\mathfrak{g}}} & \prod_m [L_{\mathfrak{g}}\Lambda_m \mathbb{1}\text{-mod}_{\mathcal{C}}] \end{array} \right)$$

where the horizontal functor is localization $L_{\mathfrak{g}}$ on each factor, and the vertical functor is extension of scalars on each factor. However, this cannot form a model for \mathcal{C} as we have simply removed too much information: the splicing data for assembling the contributions at individual closed points needs to be ‘the same almost everywhere’ in the sense that an element of the adèles is given by $(x_p) \in \prod_p \mathbb{Q}_p$ with $x_p \in \mathbb{Z}_p^{\wedge}$ for almost all p .

From another point of view the super-separated square fails to be a model because the square

$$\begin{array}{ccc} X & \longrightarrow & L_{\mathfrak{g}}\mathbb{1} \otimes X \\ \downarrow & & \downarrow \\ \prod_m [\Lambda_m \mathbb{1} \otimes X] & \longrightarrow & \prod_m [L_{\mathfrak{g}}\Lambda_m \mathbb{1} \otimes X] \end{array}$$

is usually not a homotopy pullback square.

9.D. The skeleton of the complete model

Under somewhat more restrictive hypotheses, we identify a local skeleton of the complete model $L_{\Lambda}R_{\text{wqc}}\mathcal{C}_{\text{ad}}$.

By the same proof as Lemma 9.1 we compare weak and strong qc objects.

Lemma 9.7. *For a weakly qc object X in $\mathcal{C}_{L\kappa}$, the counit $\Gamma_{\text{qc}}X \xrightarrow{\cong} X$ is an equivalence in $R_{\text{wqc}}\mathcal{C}_{L\kappa}$, so cellularization with respect to wqc coincides with that for qc. \square*

Theorem 9.8. *Let \mathcal{C} be a 1-dimensional Noetherian model category which is formally algebraic, has strict generation and so that v is faithfully flat. Then there are Quillen equivalences*

$$\mathcal{C} \simeq_Q R_{\text{wqce}}\mathcal{C}_{\text{ad}} \simeq_Q L_{\Lambda}L_{\Pi}R_{\text{wqc}}\mathcal{C}_{\text{ad}} \simeq_Q L_{\Lambda}L_{\Pi}\mathcal{C}_{\text{ad}}^{\text{qc}} \simeq_Q L_{\Lambda}\mathcal{C}_{\text{sep}}^{\text{qc}} \simeq_Q \mathcal{C}_{L\kappa}^{\text{qc}},$$

and hence the skeleton of the complete model $\mathcal{C}_{L\kappa}^{\text{qc}}$ is Quillen equivalent to \mathcal{C} .

Proof. The equivalences up to $L_{\Lambda}\mathcal{C}_{\text{sep}}^{\text{qc}}$ are exactly as in Theorem 9.5. For the remaining equivalence $L_{\Lambda}\mathcal{C}_{\text{sep}}^{\text{qc}} \simeq \mathcal{C}_{L\kappa}^{\text{qc}}$, we combine the left adjoints $L_0^{\text{m}} : R_{\text{m-mod}} \rightarrow L_0^{\text{m}} - R_{\text{m-mod}}$, to obtain

$$\ell : \prod_{\text{m}}(R_{\text{m-mod}}) \rightarrow \prod_{\text{m}}(L_0^{\text{m}} - R_{\text{m-mod}})$$

and then use the construction of Section 7.B to give a functor $\ell_* : \mathcal{C}_{\text{sep}}^{\text{qc}} \rightarrow \mathcal{C}_{L\kappa}^{\text{qc}}$ left adjoint to the inclusion. We give $\mathcal{C}_{L\kappa}^{\text{qc}}$ the right-lifted model structure as in Subsection 9.B, giving a Quillen adjunction relating $\mathcal{C}_{\text{sep}}^{\text{qc}}$ and $\mathcal{C}_{L\kappa}^{\text{qc}}$. This localizes further to give a Quillen adjunction relating $L_{\Lambda}\mathcal{C}_{\text{sep}}^{\text{qc}}$ and $\mathcal{C}_{L\kappa}^{\text{qc}}$, and this is a Quillen equivalence by applying Proposition 6.3 at the nub. \square

Part 3. Examples

In the final part of this paper where we make our results explicit in several examples, aiming to give a systematic and connected account easing comparison.

10. Abelian groups

The prototypical example is the derived category of abelian groups $\text{Ho}(\mathcal{C}) = \text{D}(\mathbb{Z})$. This applies without change if \mathbb{Z} is replaced by an arbitrary 1-dimensional commutative Noetherian domain.

The underlying category is $\mathbb{Z}\text{-mod}$. Here \mathbb{Z} is viewed as a dg ring entirely in degree 0 with zero differential. Thus $\mathbb{Z}\text{-mod}$ means the category of dg \mathbb{Z} -modules (i.e., chain complexes of abelian groups). (When we need to consider abelian groups without grading or differential, we say so explicitly.)

We make $\mathbb{Z}\text{-mod}$ into a model category using the (algebraically) projective model structure. There is a natural order-reversing homeomorphism

$$\text{Spec}(\mathbb{Z}) \xrightarrow{\cong} \text{Spc}^{\omega}(\text{D}(\mathbb{Z}))$$

where the classical algebraic prime \mathfrak{p} corresponds to the Balmer prime $\mathfrak{p}_b = \{M \mid M_{\mathfrak{p}} \simeq 0\}$ [5]. The closed points are the algebraically maximal primes (p) , and there is a unique generic point corresponding to the algebraic prime (0) , and we write $\mathbb{Q} = \mathbb{Z}_{(0)}$ for the field of fractions.

The adelic approximation theorem gives a homotopy pullback square in $\mathbb{Z}\text{-mod}$:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & \mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge \end{array}$$

We note that this square is also a strict pullback square in the underlying category of abelian groups, familiar as the classical Hasse square.

In this setting, we have both strict generization and p -adic completion functors, so we will get three different skeletons.

By Corollary 9.2, the cellular skeleton of the adelic model is

$$(\mathbb{Z}\text{-mod})_{\text{ad}}^{\text{qce}} = \Gamma \left(\begin{array}{ccc} & & \mathbb{Q}\text{-mod} \\ & & \downarrow j_* \\ & & \bullet \\ (\prod_p \mathbb{Z}_p^\wedge)\text{-mod} & \xrightarrow{\bullet} & (\mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge)\text{-mod} \end{array} \right)$$

The objects of the cellular skeleton of the adelic model are chain complexes of diagrams

$$\left[\begin{array}{ccc} & & U \\ & & \vdots \\ & & \bullet \\ & & \downarrow \\ M & \cdots \bullet \rightarrow & P \end{array} \right]$$

where U is a \mathbb{Q} -vector space, M is a module over $\prod_p \mathbb{Z}_p^\wedge$ and the information at P is an isomorphism of $\mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge$ -modules $\mathbb{Q} \otimes M \cong j_*U$.

By Theorem 9.5 the separated skeleton model is

$$(\mathbb{Z}\text{-mod})_{\text{sep}}^{\text{qcie}} = \Gamma \left(\begin{array}{ccc} & & \mathbb{Q}\text{-mod} \\ & & \downarrow j_* \\ & & \blacksquare \\ \prod_p (\mathbb{Z}_p^\wedge\text{-mod}) & \xrightarrow{\bullet} & (\mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge)\text{-mod} \end{array} \right)$$

where we have used \blacksquare to indicate the ie condition. The objects of the separated model are chain complexes of diagrams

$$\left[\begin{array}{ccc} & & U \\ & & \vdots \\ & & \blacksquare \\ & & \downarrow \\ \{M_p\} & \cdots \bullet \rightarrow & P \end{array} \right]$$

where U is a \mathbb{Q} -vector space, each M_p is a module over \mathbb{Z}_p^\wedge , and the information at P is given by a map $U \rightarrow \mathbb{Q} \otimes \prod_p M_p$ which is extension of scalars along $\mathbb{Q} \rightarrow \mathbb{Q}_p^\wedge$ on each idempotent piece.

Finally, in the language of Section 9.D, the skeleton of the complete model is

$$(\mathbb{Z}\text{-mod})_{L\kappa}^{\text{qc}} = \Gamma \left(\begin{array}{ccc} & & \mathbb{Q}\text{-mod} \\ & & \downarrow j_* \\ \prod_p L_0^p - \mathbb{Z}_p^\wedge\text{-mod} & \xrightarrow{\bullet} & (\mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge)\text{-mod} \end{array} \right)$$

The objects of the complete model are chain complexes of diagrams

$$\left[\begin{array}{ccc} & & U \\ & & \vdots \\ & & \downarrow j_* \\ \{M_p\} & \xrightarrow{\bullet} & P \end{array} \right]$$

where U is a \mathbb{Q} -vector space, M_p is an L_0^p -complete \mathbb{Z}_p^\wedge -module, and the information at P is given by an unconstrained map $j_*U \rightarrow \mathbb{Q} \otimes \prod_p L_0^p M_p$ of $\mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge$ -modules.

Remark 10.1. To give a feeling for the different models, we recommend that the reader identifies the images in the adelic, separated and complete models of $D(\mathbb{Z})$ for the abelian groups $\mathbb{Z}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \bigoplus_p \mathbb{Z}/p, \prod_p \mathbb{Z}/p$ and moreover explores moving between the various models. For example the abelian group \mathbb{Q} corresponds to the diagrams

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & \mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge \end{array} \qquad \begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \{\mathbb{Q}_p\}_p & \longrightarrow & \prod_p \mathbb{Q}_p \end{array} \qquad \begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

in the adelic, separated and complete modules respectively. Similarly the abelian group \mathbb{Q}/\mathbb{Z} corresponds to the diagrams

$$\begin{array}{ccc} \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \bigoplus_p \mathbb{Z}/p^\infty & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccc} \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \{\mathbb{Z}/p^\infty\}_p & \longrightarrow & \mathbb{Q} \otimes \prod_p \mathbb{Z}/p^\infty \end{array} \qquad \begin{array}{ccc} \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \{\Sigma \mathbb{Z}_p^\wedge\}_p & \longrightarrow & \mathbb{Q} \otimes \prod_p \Sigma \mathbb{Z}_p^\wedge \end{array}$$

in the adelic, separated and complete modules respectively.

Remark 10.2. In this case all of the rings appearing are commutative, and the equivalence is monoidal, so we may wonder about invertible modules. Over \mathbb{Z} , invertible modules are suspensions of \mathbb{Z} , but when \mathbb{Z} is replaced by a number ring R , this is more interesting: if M is an invertible module $M_{(0)} \cong R_{(0)}$, and $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^\wedge$ so that the module is specified by non-zero elements $x_{\mathfrak{p}} \in R_{(0)} \otimes R_{\mathfrak{p}}^\wedge$ for each closed point \mathfrak{p} . The module is isomorphic to R if $(x_{\mathfrak{p}})$ comes from $R_{(0)}$, so invertible modules correspond to $[R_{(0)} \otimes \prod_{\mathfrak{p}} R_{\mathfrak{p}}^\wedge]^\times / R_{(0)}^\times$ in a familiar way.

11. Quasi-coherent sheaves on a curve

Let C be an irreducible curve over an algebraically closed field k with structure sheaf $\mathcal{O} = \mathcal{O}_C$, and ring of rational functions \mathcal{K} . We take $\mathcal{C} = \mathbf{QCoh}(C)$ to be the category of (complexes of) sheaves of quasi-coherent

modules over C . The Balmer spectrum consists of the points of C , with a single generic point containing all the closed points. The Adelic Approximation Theorem states that we have a homotopy pullback of the diagram

$$\left(\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{K} \\ \downarrow & & \downarrow \\ \prod_x \mathcal{O}_x^\wedge & \longrightarrow & \mathcal{K} \otimes \prod_x \mathcal{O}_x^\wedge \end{array} \right)$$

The category $\mathbf{QCoh}(C)$ of (chain complexes of) quasi-coherent sheaves of modules has strict generalization, so we may form the skeleton of the separated model for $\mathbf{QCoh}(C)$. This is the category

$$\mathbf{QCoh}(C)_{\text{sep}}^{\text{qcie}} = \Gamma \left(\begin{array}{ccc} & & \mathcal{K}\text{-mod} \\ & & \downarrow j_* \\ \prod_x \mathcal{O}_x^\wedge\text{-mod} & \xrightarrow[\mathcal{K} \otimes]{} & \mathcal{K} \otimes \prod_x \mathcal{O}_x^\wedge\text{-mod} \end{array} \right)$$

where the product is over closed points x . We do not make the adelic or complete models explicit.

Finally, we note the connection between the Adelic Approximation Theorem and cohomology of line bundles. Tensoring the pullback square for \mathcal{O} with the line bundle $\mathcal{O}(D)$ of a divisor $D = \sum_x n_x(x)$ and taking global sections, there is an exact sequence

$$0 \longrightarrow H^0(C; \mathcal{O}(D)) \longrightarrow \mathcal{K} \times \prod_x \mathcal{O}_x^\wedge \xrightarrow{\langle i, t_x^{-n_x} \rangle} \mathcal{K} \otimes \prod_x \mathcal{O}_x^\wedge \longrightarrow H^1(C; \mathcal{O}(D)) \longrightarrow 0$$

where t_x is a function vanishing to first order at x . This recovers Weil’s residue approach to calculating cohomology as explained in [30, II.5].

Finally, we note that the Adelic Approximation Square embodies the comparison between the Cousin and residue complexes: the top row gives the Cousin complex

$$\mathbb{1} \longrightarrow L_{\mathfrak{g}}\mathbb{1} \longrightarrow [L_{\mathfrak{g}}\mathbb{1}]/\mathbb{1}$$

(which would be $\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ for abelian groups) and the Mayer-Vietoris sequence of the Hasse-Tate square is Weil’s residue complex as above (it would be $\mathbb{Z} \longrightarrow \mathbb{Q} \oplus \prod_p \mathbb{Z}_p^\wedge \longrightarrow \mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge$ for abelian groups).

12. Stable modules for the Klein 4-group

Our next example comes from modular representation theory. Let G be a finite group, and k a field with characteristic dividing the order of G . We consider the category of kG -modules. Say that two morphisms $f, g: M \rightarrow N$ are *stably homotopic* if $f - g: M \rightarrow N$ factors through a projective module. The category of kG -modules has a model structure in which the weak equivalences are the stable equivalences, the fibrations are the epimorphisms and the cofibrations are the monomorphisms [24, Theorem 2.2.12]. The homotopy category of this model category is the (big) stable module category $\underline{\text{Mod}}kG$. Passage to syzygies gives the desuspension functor in the stable module category. Tensor product over k equips the stable module category with the structure of a rigidly small-generated tensor-triangulated category with monoidal unit k .

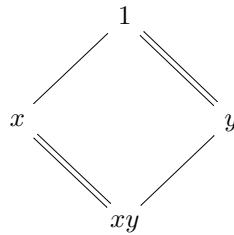
The Balmer spectrum of the small objects is the projective scheme associated to the group cohomology ring [12]:

$$\text{Spc}^\omega(\underline{\text{Mod}}kG) \cong \text{Proj } H^\bullet(G; k).$$

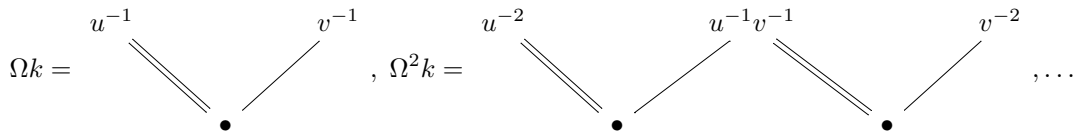
In particular, since $H^\bullet(G; k)$ is a Noetherian graded ring by Venkov’s Theorem, the Balmer spectrum is Noetherian.

When $G = E$ is the Klein 4-group $E = \langle g, h \mid g^2 = h^2 = 1, gh = hg \rangle$ and k is an algebraically closed field of characteristic 2, the cohomology ring $H^*(E; k) = \text{Ext}_{kE}^*(k, k) = k[u, v]$ is easy to deal with and the representation theory is very well understood. We take $x = g - 1$ and $y = h - 1$ and note $kE \cong k[x, y]/(x^2, y^2)$. From this point of view, the stable module category of kE is the singularity category of $k[x, y]/(x^2, y^2)$ and the BGG correspondence [11] states that the singularity category is equivalent to the bounded derived category of $\mathbb{P}^1(k) = \text{Proj}(k[u, v])$, giving another identification of the Balmer spectrum.

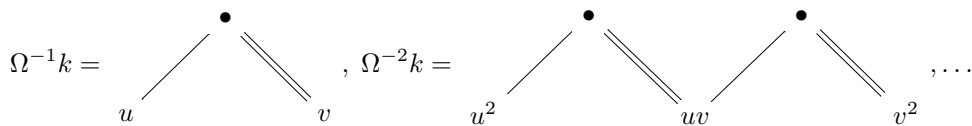
It is helpful to display kE as



where a single line refers to multiplication by x and a double line refers to multiplication by y . One sees syzygies are as follows



The labels refer to the fact that $H^n(E) = \text{Hom}(\Omega^n k, k)$ is the dual of the top vector space in the diagram: we have chosen the pairing between $k[u, v]$ and $k[u^{-1}, v^{-1}]$ by multiplying and finding the constant term. Of course $H^*(E) = \text{Sym}(E^*)$, and $\{u, v\}$ is the dual basis to $\{x, y\}$. Similarly



The labels refer to the fact that $H^n(E) = \text{Hom}(k, \Omega^{-n} k)$ is canonically isomorphic to the bottom row in the diagram. This type of labeling applies to general modules. In the stable module category, any module is equivalent to one with no free summands, and for such an M , we see that $(xy) = (x, y)^2$ acts as zero. We therefore write $V_l = \text{ann}_M(x, y)$ (l for ‘lower’) for the socle, and we may choose a complementary subspace V_u (u for ‘upper’). Multiplication by x, y gives two maps $x, y : V_u \rightarrow V_l$. We also have a canonical isomorphism $[k, M]_0 = V_l$. The group $[k, M]_*$ has the structure of a module over $[k, k]^* = \hat{H}^*(E)$, and V_l is its degree 0 part. We will constantly be passing between degree 0 information and the full graded module, and it is essential to keep clear which of the worlds each statement lives in. The syzygies of k have the property that there are endomorphisms $t : V_u \rightarrow V_u$ and $t : V_l \rightarrow V_l$ with $y = xt$ and $tx = y$ as maps $V_u \rightarrow V_l$, and it is natural to think as if $t = v/u$.

We will also need another family of modules. For $\zeta \in H^n(G, k) = \text{Hom}_{kG}(\Omega^n k, k)$, we may take $\mathbb{L}_\zeta = \ker(\zeta : \Omega^n k \rightarrow k)$, which only depends on ζ up to scalar multiplication. Since k is algebraically closed, all homogeneous elements of $H^*(E, k) = k[u, v]$ are products of degree 1 elements, so we need only consider

$\mathbb{L}_{\lambda u + \mu v}$ for $[\lambda : \mu] \in \mathbb{P}^1(k)$. The module k has support $\mathbb{P}^1(k)$ and for closed points ζ , the module \mathbb{L}_ζ has support $\{\zeta\}$.

Nullifying \mathbb{L}_ζ has the effect of inverting ζ in homotopy. Now $[k, k]_* = \hat{H}^*(E; k)$, and the norm sequence of $H^*(E)$ -modules is the short exact sequence

$$0 \longrightarrow H^*(E) \longrightarrow \hat{H}^*(E) \longrightarrow \Sigma_{-1}H_*(E) \longrightarrow 0,$$

where Σ_n is the cohomological suspension $(\Sigma_n M)^c = M^{c-n}$. Since $H_*(E)$ is torsion, this shows that inverting ζ makes $H^*(E) \longrightarrow \hat{H}^*(E)$ into an isomorphism. Localizing at the prime (ζ) involves nullifying all \mathbb{L}_ψ with $\psi \neq \zeta$ and the generic localization $L_{\mathfrak{g}}k$ involves nullifying all \mathbb{L}_ζ . Thus

$$[k, L_\zeta k]_* = k[u, v]_{(\zeta)}, [k, L_{\mathfrak{g}}k]_* = k[u, v]_{(0)}.$$

Note here that $k[u, v]_{(0)}$ is $k(t)$ in degree 0 with $t = v/u$ and any non-zero element $\zeta \in H^1(E)$ is a periodicity element. Similarly in $k[u, v]_{(\zeta)}$, except that the periodicity element is any element not a multiple of ζ .

In fact we can see in both cases how to realize these modules, by taking the direct limit along multiples of maps $\zeta \in H^1(E)$. It is convenient to think of $\zeta : k \longrightarrow \Omega^{-1}k$ and so forth, so using the identification of the lower module with the homotopy, we have

$$L_{\mathfrak{g}}k = \left(\begin{array}{c} k(t) \\ x \left(\begin{array}{cc} 1 & y \end{array} \right) t \\ k(t) \end{array} \right)$$

Of course it is natural to think of $k(t)$ the ring of rational functions on $\mathbb{P}^1(k)$, and similarly, for a closed point $\zeta \in \mathbb{P}^1(k)$ we can describe $L_\zeta k$ using the rational functions regular at ζ .

$$L_\zeta k = \left(\begin{array}{c} k[u, v]_{(\zeta)} \\ x \left(\begin{array}{cc} 1 & y \end{array} \right) t \\ k[u, v]_{(\zeta)} \end{array} \right)$$

Next, we may calculate the derived completions in general by the formula

$$\Lambda_\zeta M = \text{holim}_{\leftarrow s} M/\zeta^s.$$

Picking $\zeta = u$ for definiteness, one finds

$$[k, \Lambda_u k]_* = k[u, v, v^{-1}]_u^\wedge,$$

where the inversion of v could be replaced by the inversion of any other degree 1 element (except for multiples of u). We note that the degree 0 part of this is the completed stalk of the point ∞ (defined by $u = 0$):

$$[k, \Lambda_u k]_0 = [\mathcal{O}_{\mathbb{P}^1(k)}]_\infty^\wedge.$$

Continuing to the completion we want, we have the usual splitting

$$(L_{\mathfrak{g}}k)/k \simeq \bigoplus_{\zeta} (k[1/\zeta])/k$$

and hence

$$\Lambda_0 M := \text{Hom}(L_{\mathfrak{g}}k/k, M) \simeq \prod_{\zeta} \Lambda_{\zeta} M.$$

The Adelic Approximation Theorem states that there is a homotopy pullback in the category $\underline{\text{Mod}}(kE)$

$$\begin{array}{ccc} k & \longrightarrow & L_{\mathfrak{g}}k \\ \downarrow & & \downarrow \\ \Lambda_0 k & \longrightarrow & L_{\mathfrak{g}}\Lambda_0 k \end{array}$$

A localization of this is considered in relation to the Chromatic Splitting Conjecture in [8, Example 3.18].

Applying $[k, \cdot]_*$ we obtain the square

$$\begin{array}{ccc} \hat{H}^*(E) & \longrightarrow & k(t)[u, u^{-1}] \\ \downarrow & & \downarrow \\ \prod_{\zeta} k[u, v, (\zeta')^{-1}]_{\zeta}^{\wedge} & \longrightarrow & (\prod_{\zeta} k[u, v, (\zeta')^{-1}]_{\zeta}^{\wedge})_{(0)} \end{array}$$

where ζ' is any element of $H^1(E)$ not a multiple of ζ . The associated Mayer-Vietoris sequence of this square is the residue complex for calculating the cohomology of $\mathbb{P}^1(k)$ with coefficients in twists of the structure sheaf as in Section 11.

One can then use the above splitting to build an adelic, separated and complete model.

13. \mathbb{T} -equivariant rational stable homotopy theory

We finish the examples with a discussion of the case that motivated this work, namely the category of \mathbb{T} -equivariant rational cohomology theories, where \mathbb{T} is the circle group.

For a general compact Lie group G , we say that an inclusion $K \subseteq H$ of subgroups of G is *cotoral* if K is normal in H and H/K is a torus. The Balmer spectrum consists of the conjugacy classes of subgroups under cotoral inclusion [16].

In particular, the Balmer spectrum of $\mathbf{Sp}_{\mathbb{Q}}^{\mathbb{T}}$ is

$$\text{Spc}^{\omega}(\mathbf{Sp}_{\mathbb{Q}}^{\mathbb{T}}) = \begin{array}{cccc} & & \mathbb{T} & \\ & \nearrow & \uparrow & \nwarrow \\ \cdots & C_4 & C_3 & C_2 & C_1 \end{array}$$

where C_i is the cyclic group of order i . The diagram S_{ad} coming from the Adelic Approximation Theorem is then

$$\begin{array}{ccc} & & L_{\mathbb{T}}\mathbb{S} \\ & & \downarrow \\ \prod_n \Lambda_{C_n}\mathbb{S} & \longrightarrow & L_{\mathbb{T}}\prod_n \Lambda_{C_n}\mathbb{S} . \end{array}$$

Writing \mathcal{F} for the family of finite subgroups, this can be written as

$$\mathbb{S}_{\text{ad}} = \left(\begin{array}{ccc} & \tilde{E}\mathcal{F} & \\ & \downarrow & \\ \prod_n DE\langle n \rangle_+ & \longrightarrow & \tilde{E}\mathcal{F} \wedge \prod_n DE\langle n \rangle_+ \end{array} \right) \simeq \left(\begin{array}{ccc} & \tilde{E}\mathcal{F} & \\ & \downarrow & \\ DEF_+ & \longrightarrow & \tilde{E}\mathcal{F} \wedge DEF_+ \end{array} \right)$$

which is the usual \mathcal{F} -Tate square for rational \mathbb{T} -equivariant homotopy theory [14] (where $E\langle n \rangle$ is the mapping cone of $E[\subset C_n]_+ \rightarrow E[\subseteq C_n]_+$).

The adelic model has bifibrant objects

$$\left(\mathbb{S}\mathbb{P}_{\mathbb{Q}}^{\mathbb{T}} \right)_{\text{ad}}^{\text{wqce}} = \Gamma \left(\begin{array}{ccc} & (\tilde{E}\mathcal{F})\text{-mod}_{\mathbb{S}\mathbb{P}_{\mathbb{Q}}^{\mathbb{T}}} & \\ & \downarrow \circlearrowleft & \\ (\prod_n DE\langle n \rangle_+)\text{-mod}_{\mathbb{S}\mathbb{P}_{\mathbb{Q}}^{\mathbb{T}}} & \dashrightarrow & (\tilde{E}\mathcal{F} \wedge \prod_n DE\langle n \rangle_+)\text{-mod}_{\mathbb{S}\mathbb{P}_{\mathbb{Q}}^{\mathbb{T}}} \end{array} \right)$$

This model is the first step in [21] to the algebraic model for rational \mathbb{T} -equivariant spectra. One then shows that taking \mathbb{T} -fixed points termwise gives an equivalence, and the resulting ring spectra are formal, with homotopy

$$\mathbb{S}_{\text{ad}}^{\text{alg}} = \left(\begin{array}{ccc} & \mathbb{Q} & \\ & \downarrow & \\ \prod_n \mathbb{Q}[c] & \longrightarrow & \mathcal{E}^{-1} \prod_n \mathbb{Q}[c] \end{array} \right)$$

We note that there is no single graded ring giving rise to this adelic diagram, so the use of *diagrams* of rings is essential. Taking modules over this diagram of rings, we obtain the homotopical version of the standard algebraic model

$$\left(\mathbb{S}_{\text{ad}}^{\text{alg}}\text{-mod} \right)^{\text{wqce}} := \Gamma \left(\begin{array}{ccc} & \mathbb{Q}\text{-mod} & \\ & \downarrow \circlearrowleft & \\ (\prod_n \mathbb{Q}[c])\text{-mod} & \dashrightarrow & (\mathcal{E}^{-1} \prod_n \mathbb{Q}[c])\text{-mod} \end{array} \right)$$

The generic localization is inverting the set \mathcal{E} generated by the Euler classes $e(z^n)$ (which is c at subgroups of C_n and 1 otherwise). It is therefore strictly smashing, so we can form skeletons of the adelic, separated and complete models. The proof that the separated and complete diagrams give models for rational \mathbb{T} -spectra appears here for the first time. The standard model is the cellular skeleton of the algebraic model

$$dg\mathcal{A}(\mathbb{T}) = \left(\mathbb{S}_{\text{ad}}^{\text{alg}}\text{-mod} \right)^{\text{qce}} := \Gamma \left(\begin{array}{ccc} & \mathbb{Q}\text{-mod} & \\ & \downarrow \bullet & \\ (\prod_n \mathbb{Q}[c])\text{-mod} & \dashrightarrow & (\mathcal{E}^{-1} \prod_n \mathbb{Q}[c])\text{-mod} \end{array} \right).$$

This is the category of differential graded objects of the standard abelian category $\mathcal{A}(\mathbb{T})$ of [14]. (The category $\mathcal{A}(\mathbb{T}) = \mathcal{A}_{\text{ad}}(\mathbb{T})$ is obtained by interpreting ‘module’ in the classical restricted sense of the abelian category of graded modules with no differential.) The skeleton of the separated model is

$$dg\mathcal{A}_{\text{sep}}(\mathbb{T}) = (\mathbb{S}_{\text{ad}}^{\text{alg}}\text{-mod})_{\text{sep}}^{\text{qcie}} := \Gamma \left(\begin{array}{c} \mathbb{Q}\text{-mod} \\ \downarrow \blacksquare \\ \prod_n (\mathbb{Q}[c]\text{-mod}) \xrightarrow{\bullet} (\mathcal{E}^{-1} \prod_n \mathbb{Q}[c]\text{-mod}) \end{array} \right).$$

This is the category of differential graded objects of the separated abelian category $\mathcal{A}_{\text{sep}}(\mathbb{T})$ defined as for $\mathcal{A}(\mathbb{T})$. The skeleton of the complete model is

$$dg\mathcal{A}_{L\kappa}(\mathbb{T}) = (\mathbb{S}_{\text{ad}}^{\text{alg}}\text{-mod})_{L\kappa}^{\text{qce}} := \Gamma \left(\begin{array}{c} \mathbb{Q}\text{-mod} \\ \downarrow \\ \prod_n (L_0^{(c)} - \mathbb{Q}[c]\text{-mod}) \xrightarrow{\bullet} (\mathcal{E}^{-1} \prod_n \mathbb{Q}[c]\text{-mod}) \end{array} \right).$$

This is the category of differential graded objects of the complete abelian category $\mathcal{A}_{L\kappa}(\mathbb{T})$ defined as for $\mathcal{A}(\mathbb{T})$.

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